

**MATH 10**  
**ASSIGNMENT 18: SERIES II**  
MAR 8, 2020

SERIES

Recall: given a sequence  $a_n$ , we define

$$\sum_{i=1}^{\infty} a_i = \lim S_n, \quad \text{where}$$
$$S_n = a_1 + \cdots + a_n = \sum_{i=1}^n a_i$$

(if this limit exists; otherwise we say that the series diverges and expression  $\sum_{i=1}^{\infty} a_i$  is meaningless). For example:

$$1 + r + r^2 + \cdots = \sum_{i=0}^{\infty} r^i = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}, \quad |r| < 1$$

(this series is called the *geometric series*).

Note that it is quite possible that the sequence  $a_n$  converges but the sequence  $S_n$  of partial sums does not converge and thus the series  $\sum_{i=1}^{\infty} a_i$  diverges!!

In the last HW, we have proved the following facts.

**Theorem.**

1. If a series  $\sum a_n$  converges, then  $\lim a_n = 0$ . (Converse is not true: even if  $\lim a_n = 0$ , the series may diverge).
2. If  $0 \leq a_n \leq b_n$ , and  $\sum b_n$  converges, then  $\sum a_n$  also converges.

In fact, there is a more general result:

**Theorem** (Comparison test). If  $a_n, b_n$  are sequences such that  $b_n \geq 0$ ,  $|a_n| \leq b_n$  and the series  $\sum_1^{\infty} b_n$  converges, then  $\sum_1^{\infty} a_n$  also converges.

The proof of this result will be given later.

HOMEWORK

1. A tortoise is moving on the plane starting at the origin and then going 1 unit along the positive direction of  $x$  axis; then turning  $90^\circ$  to the left and going for 0.9 units, then turning  $90^\circ$  to the left and going for  $(0.9)^2$  units, then....
  - (a) Show that if we consider the plane as the complex plane  $\mathbb{C}$ , then the position of the tortoise after  $n$  steps will be at the point  $1 + r + r^2 + \cdots + r^{n-1}$ , where  $r = 0.9i$ .
  - (b) Find where the tortoise will end up in the limit, after infinitely many steps.
2. Let  $a_n$  be a sequence such that  $r = \lim \frac{|a_{n+1}|}{|a_n|}$  exists.
  - (a) Show that if  $r > 1$ , then  $\lim a_n$  does not exist, and therefore  $\sum_1^{\infty} a_n$  diverges (compare with Problem 1 from previous HW).
  - (b) Prove that if  $r < 1$ , then the series  $\sum_1^{\infty} a_n$  converges. [Hint: compare with geometric series.]
  - (c) Give examples showing that if  $r = 1$ , then series  $\sum_1^{\infty} a_n$  may converge or diverge.This is known as the *ratio test* for series convergence.
3. Prove that for any  $x \in \mathbb{C}$ , the series

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges. [Hint: use the previous problem.]

4. Let  $E(x)$  be as defined in the previous problem. Prove that then  $E(x+y) = E(x)E(y)$ . [Hint: both sides can be written as “double series”  $\sum a_{m,n}x^ny^m$ . You can use without a proof that in all the series involved, rearranging the terms in any order will not affect the value of the series.]
5. Let  $e = \sum_{n=0}^{\infty} \frac{1}{n!} = E(1)$ . Prove that then  $E(x) = e^x$ :
- (a) For all integer  $x$
  - (b) For all rational  $x = p/q$
  - \* (c) For all real  $x$