MATH 10
ASSIGNMENT 14: LIMITS CONTINUED

FEB 3, 2020

Today we will be discussing limits of sequences of real numbers (but many of the results could be generalized to sequences of points in a plane, or in fact to sequences in any metric space).

Recall the definition of limit:

Definition. A number a is called the *limit* of sequence a_n (notation: $a = \lim a_n$) if for any $\varepsilon > 0$, all terms of the sequence starting with some index N will be in the interval $(a - \varepsilon, a + \varepsilon)$: for any $n \geq N$, $|a_n - a| < \varepsilon$.

However, usually one computes limits not by using this definition but rather using the following *limit laws*:

Theorem 1. *Let sequences a_n, b_n be such that $\lim a_n = A$, $\lim b_n = B$. Then:*

1. $\lim(a_n + b_n) = A + B$
2. $\lim(a_n b_n) = AB$
3. $\lim(a_n/b_n) = A/B$ *(only holds if $B \neq 0$).*

In addition, there is also the following result:

Theorem 2. *If $a_n \geq 0$, $\lim a_n = 0$, and $|b_n| \leq a_n$, then $\lim b_n = 0$.*

The following limits are useful:

$$\text{If } a_n \text{ is a constant sequence: } a_n = c \text{ for all } n, \text{ then } \lim a_n = c$$
$$\lim \tfrac{1}{n} = 0$$
$$\text{If } |r| < 1, \text{ then } \lim r^n = 0$$

Sometimes in order to use these rules, some tricks are necessary. For example, one can not compute the limit $\lim \frac{n+2}{2n+3}$ direclty, as $\lim(n+2)$ does not exist. However, a simple trick allows one to use the quotient rule:

$$\lim \frac{n+2}{2n+3} = \lim \frac{1 + \frac{2}{n}}{2 + \frac{3}{n}} = \frac{1+0}{2+0} = \frac{1}{2}$$

Using these rules, we had computed the following important limit

$$1 + r + r^2 + \cdots = \lim(1 + r + \cdots + r^n) = \lim \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1-r}, \qquad |r| < 1 \tag{1}$$

1. Compute the following limits.
(a) $\lim \frac{2n^2+n+1}{n^2+3}$
(b) $\lim \frac{n^2+15n}{n^3}$
(c) $\lim \frac{2^n+1}{3^n}$
*(d) $\lim \frac{n}{2^n}$ [Show first that for $n \geq 3$, one has $a_{n+1}/a_n \leq 2/3$. Deduce then that $a_n \leq C(2/3)^n$ for some constant C.]

2. Prove that a sequence that has a limit must be bounded. [Hint: if $\lim a_n = A$, then starting from some moment, all terms of the sequence are $\leq A + 1$.]

***3.** Prove Theorem 2.

4. Prove that if $|b_n| \leq 2$, and $\lim a_n = 0$, then $\lim a_n b_n = 0$.

5. Consider the sequence defined by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \tfrac{2}{a_n}\right) \tag{2}$$

(a) Use a calculator or a computer to compute the first 5 terms. Does it indeed look like the sequence is convergent? [You are not required to give a rigorous proof that it is convergent.]

(b) Assuming that it does converge, can you guess what the value of the limit is? [Hint: if this sequence is convergent, then the limit A satisfies $A = \frac{1}{2}(A + \frac{2}{A})$.]
(c) Can you modify (2) to get a sequence that computes $\sqrt{3}$?

6. Consider the sequence given by $x_1 = 1$, $x_{n+1} = \frac{1}{1+x_n}$.
(a) Compute first 3 terms of this sequence.
(b) Prove that if the limit exists, it satisfies $X = \frac{1}{1+X}$.
(c) Assuming that the limit exists, find it.

7. Consider the infinite decimal

$$x = 0.17171717\ldots$$

(a) Show that this decimal can be written as a sum of an infinite geometric progression.
(b) Show that x is a rational number.
(c) Is it true that any periodic infinite decimal is rational? Is the converse true?

8. Here is a classic problem.

Katerina's and Rachel's houses are 3 miles apart. At exactly 4pm, Katerina and Rachel start walking, each starting at her own house and walking to her friend's one. Each of them walks at 3 miles per hour. Katerina has a dog, and as soon as they start walking, the dog runs towards Rachel at 6 miles per hour; reaching Rachel, he immediately turns back and runs to Katerina; reaching Katerina, he again starts running to Rachel, and so on, until the friends meet. How many miles will the dog run overall?

There are two ways to solve it. The smart way is to determine when the friends meet, and since we know dog's speed, we can easily compute the distance. A longer way is to find out how many miles he run on his first run (until he reaches Rachel), then on the second, and so on, and then add it all up — which requires summing up a geometric progression. Can you do the problem both ways and check that the answers match?

9. Antonine the ant is walking on the real line. It's position x as a function of time t is given by $x(t) = 4t^3 - 2t^2 + 3t + 5$.
(a) What is the average speed of Antonine between the instants $t = 0$ and $t = 1$? In what direction was Antonine moving in this interval (on average)?
(b) Consider the sequence $v_n(t)$ = "average speed of Antonine the ant between time t and time $t + 1/n$". Does the sequence converge? If yes, what is the limit? Can you give a physical interpretation for the limit?

10. Consider the parabola given by $y(x) = x^2 + 3x + 4$.
(a) Find the line that intersects the parabola at the point with $x = 0$ and $x = 1$. Draw this line and the parabola.
(b) Draw the sequence of lines that intersect the parabola at the points with $x = 0$ and $x = 1/n$ for a few values of n. What is the geometrical interpretation of the limit of this sequence?
(c) Find the line that is tangent to the parabola at the point $(x, y(x))$.