MATH 10
ASSIGNMENT 13: LIMITS
JANUARY 26, 2020

Limits

We say that a sequnce a_n has limit A if, as n increases, terms of the sequence get closer and closer to A.

This definition is not very precise. For example, the terms of sequence $a_n = 1/n$ get closer and closer to 0, so one expects that the limit is 0. On the other hand, it is also true that they get closer and closer to -1. So the words "closer and closer" is not a good way to express what we mean.

A better way to say this is as follows.

Definition. A set U is called a *trap* for the sequence a_n if, starting with some index N, all terms of the sequence are in this set:

$$\exists N : \quad \forall n \geq N : \ a_n \in U$$

Note that it is not the same as "infinitely many terms of the sequence are in this set".

Now we can give a rigorous definition of a limit.

Definition. A number A is called the *limit* of sequence a_n (notation: $A = \lim a_n$) if for any $\varepsilon > 0$, the neighborhood $B_\varepsilon(A) = \{x \mid d(x, A) < \varepsilon\}$ is a trap for the sequence a_n.

For example, when we say that for a sequence $a_n \in \mathbb{R}$, $\lim a_n = 3$, it means:

there is an index N such that for all $n \geq N$ we will have $a_n \in (2.99, 3.01)$,
there is an index N' (possibly different) such that for all $n \geq N'$ we will have $a_n \in (2.999, 3.001)$
there is an index N'' such that for all $n \geq N''$ we will have $a_n \in (3 - 0.0000001, 3 + 0.0000001)$
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1. Consider the sequence $a_n = 1/n$ ($a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$,...).
   (a) Fill in the blanks in each of the statements below:
   – For all $n \geq$ ____, $|a_n| < 0.1$
   – For all $n \geq$ ____, $|a_n| < 0.001$
   – For all $n \geq$ ____, $|a_n| < 0.00017$
   Each one of these assertions implies that a certain set is a trap for the sequence $a_n = 1/n$. Write down these three sets.
   (b) Show that $\lim a_n = 0$.

2. Prove that $\lim \frac{1}{n(n+1)} = 0$ (hint: $\frac{1}{n(n+1)} < \frac{1}{n}$).

3. Find the limits of the following sequences if they exist:
   (a) $a_n = \frac{1}{n^2}$
   (b) $a_n = \frac{1}{2^n}$
   (c) $a_n = n$

4. Explain why the number 1 is NOT a limit of the sequence $(-1)^n$.

5. (a) Show that the limit of a sequence (if exists) is an accumulation point.
   (b) Show that converse is not necessarily true: an accumulation point does not have to be a lmit.
   (c) Show that if a sequence has two different accumulation points C, C', then it cannot have a limit.

6. Show that the set of accumulation points of a sequence is closed.

7. (a) Let S be a closed set (i.e., a set that contains all of its accumulation points, see problem **4**.c) in Homework 13) and a_n a sequence such that $a_n \in S$ for any n. Prove that if the limit $\lim a_n$ exists, it must be also in S.
   (b) Let $a_n \geq 0$ for all n. Prove that then $\lim a_n \geq 0$ (assuming it exists).
   (c) Let $a_n > 0$ for all n. Is it true that then $\lim a_n > 0$ (assuming it exists)?