MATH 10

ASSIGNMENT 12: ACCUMULATION POINTS

JAN 12, 2020

Review of last class

Given a point $x \in X$ and a positive real number ε, we define ε-neighborhood of x by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

For $X = \mathbb{R}$, neighborhoods are just open intervals: $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$

If $S \subset X$, denote by S' the complement of S. Then, for any $x \in X$, we can have one of three possibilities:

1. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S (in paritcular, this implies that $x \in S$). Such points are called *interior points* of S; set of interior points is denoted by $\mathrm{Int}(S)$.

2. There is a neighborhood $B_\varepsilon(x)$ which is completely inside S' (in paritcular, this implies that $x \in S'$). Thus, $x \in \mathrm{Int}(S')$.

3. Any neighborhood of x contains points from S and points from S' (in this case, we could have $x \in S$ or $x \in S'$). Set of such points is called the *boundary* of S and denoted ∂S.

Definition. A set S is called *open* if every point $x \in S$ is an interior point: $S = \mathrm{Int}(S)$.
A set S is called closed if $\partial S \subset S$.

Part of last week homework was to show that a set S is open if and only if its complement is closed.

Accumulation points

As before, all our constructions take place in some metric space X (such as $\mathbb{R}$, $\mathbb{R}^2$, etc).

Definition. Let x_n be a sequence of points in X. We say that $A \in X$ is an *accumulation point* of x_n if each neighborhood of A contains infinitely many terms of the sequence.

For example, if $x_n = 1/n \in \mathbb{R}$, then point $A = 0$ is an accumulation point: in any neigborhood $(-\varepsilon, \varepsilon)$ there are infinitely many terms of the sequence (namely, all x_n with $n > 1/\varepsilon$).

Homework

1. Find all accumulation points of the following sequences:
   (a) Sequence $x_n = \frac{1}{n}$
   (b) Sequence $a_n = (-1)^n + \frac{1}{n}$: $a_1 = -1 + 1 = 0$, $a_2 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_3 = -1 + \frac{1}{3} = -\frac{2}{3}$, $\ldots$, $a_{100} = 1 + \frac{1}{100}$, $a_{101} = -1 + \frac{1}{101}$,...
   (c) $x_n = n + 1/n$.

2. Is it possible to construct a sequence a_n of real numbers so that the set of its accumulation points is
   (a) Set consisting of just two points $\{0, 1\}$
   (b) Empty set
   (c) Interval $[0, 1]$ (Hint: make your sequence contain all rational numbers in this interval).
   (d) Set $\mathbb{N} = \{1, 2, 3, \ldots\}$.

3. (a) Let set S be the set of all irrational numbers satisfying inequality $0 < x < 1$. Show that one can construct a sequence $x_n \in S$ which has $A = 1$ as one of its accumulation points.
   (b) Show that for any set S and a point $A \in \partial S$, one can choose a sequence of elements of S which has A as one of its accumulation points.

4. (a) Let $S = [0, 1] \subset \mathbb{R}$. Is it possible to construct a sequence $x_n \in S$ such that it has $A = 1.1$ as an accumulation point?
   (b) Show that for any sequence $x_n \in [0, 1]$, all accumulation points of this sequence (if any) are in $[0, 1]$
   (c) Show that if S is a closed set (and thus its complement is an open set), then for any sequence of elements of S, all its accumulation points are in S.

(d) Give a counterexample to show that the statement of the previous part may fail if we do not assume that S is closed.

***5.** Is it possible to construct a sequence a_n so that the set of its accumulation points is the set of all rational numbers?

If possible, give a construction; if impossible, try to explain why.