MATH 10
ASSIGNMENT 5: MATRIX INVERSE
OCTOBER 20, 2019

Rank

Definition. Let A be an $[m, n]$ matrix. Then the *rank* of A, $rank(A)$ is the number of nonzero rows of the matrix when it is put in row echelon form. Similarly, for a linear function f between vector spaces V and W, $f : V \to W$, the *rank* of f is defined as the *rank* of the matrix corresponding to f in given bases in V and W.

Again, this number can be shown to be independent of the choice of bases and of how the matrix is brought into row echelon form. With this definition, we can rewrite the theorem from the last class in a more elegant way. Let us denote the dimension of a vector space V by $dim(V)$.

Theorem 1. *Let $f : V \to W$ be a linear function. Suppose that, for a given $w \in W$, the linear equation $w = f(v)$ has solutions $v \in V$. Then the dimension d of the space of solutions is given by*

$$d = dim(V) - rank(f)$$

Matrix Inverse

Recall that every linear function $f : \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = ax$ for some $a \in \mathbb{R}$. Then we know that, in the simple case where $a \neq 0$, $b \neq 0$, the linear equation $f(x) = b$ has the unique solution $x = a^{-1}b$. Now we want to find a similar expression for cases when the domain and the codomain of the funtion have dimension bigger than 1. We saw that, in this case, there is a useful correspondence between linear functions and matrices. So we are led to consider the following concept:

Definition. Given matrices $A \in \mathbb{M}[m, n]$, $B \in \mathbb{M}[n, m]$, we say that B is an *inverse* of A (or A is an inverse of B) if $AB = I_m$ and $BA = I_n$.

Here, I_n denotes the *identity matrix* of order n. This is the $[n, n]$ square matrix whose elements $I_{i,j}$ are 1 if $i = j$ and 0 otherwise:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Note that this matrix indeed works like an identity for matrix multiplication: $AI_n = A, \forall A \in \mathbb{M}[m, n]$, $I_nB = B, \forall B \in \mathbb{M}[n, p]$.

In the homework, you are going to prove the following:

Theorem 2. *Let $A \in \mathbb{M}[m, n]$. Then A has an inverse if and only if $n = m$ and $rank(A) = m$. Moreover, if an inverse exists, it is unique.*

Thus we can unambiguously use the notation A^{-1} for *the* inverse of a matrix.

There are many methods to compute the inverse of a matrix. One you can use is the following. For a given invertible $A \in \mathbb{M}[n, n]$, write the augmented matrix $A|I_n$. By elementary row operations (the operations you would use to bring a matrix to row echelon form), bring the augmented matrix to the form $I_n|B$. Then $B = A^{-1}$.

Homework

1. *Invertibility criterion*

(a) Let $f : A \to B$ be a general function between to sets. Show that f is invertible (ie., there exists a function $f^{-1} : B \to A$ such that $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$, where $id_C : C \to C$ is defined by $id_C(x) = x, \forall x \in C$) if and only if it is bijective (both injective and surjective).

(b) Now take $f : V \to W$ to be a linear function between vector spaces V and W. Argue (no need for a rigorous proof) that, if f is bijective, then $dim(V) = dim(W)$.

(c) Now show that, if f is injective, then $rank(f) = dim(V)$. *(Hint: think of theorem 1.)*

(d) Now finish showing that a linear function $f : V \to W$ is invertible if and only if $dim(W) = dim(V)$ and $rank(f) = dim(V)$. *(Hint: show surjectivity by considering the system of equations corresponding to $f(v) = w$ when the matrix corresponding to the linear function f is in row echelon form.)*

(e) Show that, if a linear function $f : V \to W$ is invertible, then $f^{-1} : W \to V$ is also a linear function.

(f) Show that, if F and G are the matrices corresponding to the linear functions f and its inverse, then $G = F^{-1}$. *(Hint: remember problem 4 of homework 2.)*

(g) Show that, if B is an inverse of a given matrix A and C is also an inverse of A, then $B = C$.

(h) Use all the above to prove theorem 2.

2. Determine if each one of these matrices is invertible. If yes, find the inverse.

(a)

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{bmatrix}.$$

3. Consider the system of equations

$$\begin{aligned} 2x_2 + 4x_3 &= y_1 \\ 2x_1 + 4x_2 + 2x_3 &= y_2 \\ 3x_1 + 3x_2 + x_3 &= y_3 \end{aligned}$$

(a) Does the system have a solution for arbitrary y_1, y_2, y_3? When it has a solution, what is it?

(b) Is the matrix of coefficients invertible? If yes, what is the inverse?

(c) Can you use the inverse of the matrix of coefficients to write the general solution for arbitrary y_1, y_2, y_3?