MATH 10
ASSIGNMENT 3: SYSTEMS OF LINEAR EQUATIONS
SEPTEMBER 29, 2019

Linear equations

A *linear equation* is an equality of the form $\vec{y} = f(\vec{x})$, where f is a given linear function and $\vec{y}$ is a given vector. The set of solutions is the set of vectors $\vec{x}$ which satisfy the equation.

From the previous homework, we know that, in the case $f : \mathbb{R} \to \mathbb{R}$, the equation can be written as $y = ax$, where $a, y \in \mathbb{R}$ are given and $x \in \mathbb{R}$ is to be found. We have that, if $y = 0$, then $x = 0$ is a solution. If additionally $a \neq 0$, this is the unique solution. On the other hand, if $a = 0$, then actually any real number x is a solution. In case $y \neq 0$ and $a \neq 0$, there is a unique solution $x = y/a$, while if $y \neq 0$ and $a = 0$ then there is no solution.

As soon as we exchange the domain and/or codomain of f by other vector spaces (like vectors in the plane), this classification becomes more complicated. To start, remember from the last homework that, using basis vectors in the domain and in the codomain, the equation becomes something of the form $\mathbf{y} = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \ A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

for some given matrices $\mathbf{y}$ and A. We will start to study methods of finding $\mathbf{x}$ before we arrive at general results like in the $f : \mathbb{R} \to \mathbb{R}$ case.

A good way of looking at the equation $\mathbf{y} = A\mathbf{x}$ is performing the matrix multiplication to find

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}.$$

This can then be seen as a *system of linear equations.*

Systems of linear equations

Making the discussion more concrete, a system of linear equations would look like

$$2x_1 + x_2 + 3x_3 = 2$$
$$4x_1 - 7x_2 + 5x_3 = 1$$

This system of linear equations corresponds to

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -7 & 5 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} 2 \\ 1. \end{bmatrix}$$

The *augmented* matrix is formed by putting these together:

$$A|\mathbf{y} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 2 \\ 4 & -7 & 5 & 1 \end{array}\right].$$

Elementary row operations

The main idea of solving an arbitrary system of linear equations is by transforming it to a simpler form. Transformation should not change the set of solutions. To do this, we will use the following elementary operations:

- Exchange two equations (= two rows of the augmented matrix)
- Multiply both sides of an equation (= one row of augmented matrix) by a non-zero number
- Add to one equation a multiple of another.

Applying these operations to bring your matrix to a simpler form is called *row reduction*, or *Gaussian elimination*

Simple example

$$
\begin{aligned}
x_1 - 2x_2 + 2x_3 &= 5 \\
x_1 - x_2 &= -1 \\
-x_1 + x_2 + x_3 &= 5
\end{aligned}
$$

The augmented matrix is

$$
\left[\begin{array}{ccc|c}
1 & -2 & 2 & 5 \\
1 & -1 & 0 & -1 \\
-1 & 1 & 1 & 5
\end{array}\right]
$$

Using row operations, we can bring it to the form

$$
\left[\begin{array}{ccc|c}
1 & -2 & 2 & 5 \\
0 & 1 & -2 & -6 \\
0 & 0 & 1 & 4
\end{array}\right]
$$

so the solution is

$$
\begin{aligned}
x_3 &= 4 \\
x_2 &= -6 + 2x_3 = 2 \\
x_1 &= 5 - 2x_3 + 2x_2 = 1
\end{aligned}
$$

Row echelon form

In general, using row operations, every system can be brought to a form where each row begins with some number of zeroes, and each next row has more zeroes than the previous one:

$$
\left[\begin{array}{ccccccc|c}
X & * & * & * & * & * & * & * \\
0 & 0 & 0 & X & * & * & * & * \\
0 & 0 & 0 & 0 & X & * & * & *
\end{array}\right]
$$

(here X's stand for non-zero entries).

To solve such a system, we do the following:

- Variables corresponding to columns with X's in them are called pivot variables; the remaining ones are called free variables.
- Values for free variables can be chosen arbitrarily. Values for pivot variables are then uniquely determined from the equations.

For example, in the system

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 5 \\
x_2 + 3x_3 &= 6
\end{aligned}
$$

varaibles x_1, x_2 are pivot, and variable x_3 is free, so we can solve it by letting $x_3 = t$, and then

$$
\begin{aligned}
x_2 &= 6 - 3x_3 = 6 - 3t \\
x_1 &= 5 - x_2 - x_3 = -1 + 2t
\end{aligned}
$$

Homework

1. Solve the following system of equations

$$\begin{aligned} w + x + y + z &= 6 \\ w + y + z &= 4 \\ w + y &= 2 \end{aligned}$$

2. Solve the system of equations with the following matrix

$$\left[\begin{array}{ccccc} 2 & -1 & 3 & | & 2 \\ 1 & 4 & 0 & | & -1 \\ 2 & 6 & -1 & | & 5 \end{array}\right]$$

3. Solve the following system of equations

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + 8x_3 &= 4 \end{aligned}$$

4. Consider the system of equations

$$\begin{aligned} 3x - y + 2z &= b_1 \\ 2x + y + z &= b_2 \\ x - 7y + 2z &= b_3 \end{aligned}$$

(a) If $b_1 = b_2 = b_3 = 0$, find all solutions
(b) For which triples b_1, b_2, b_3 does it have a solution?

5. Consider a system of 4 equations in 5 variables.
(a) Show that if the right-hand side is zero, then this system must have a non-zero solution.
(b) Is it true if the right-hand side is non-zero?