MATH 10
ASSIGNMENT 2: INTRODUCING $\mathbb{R}^n$ AND MATRICES
SEPTEMBER 21, 2019

Vectors

We saw that we can write all vectors $\vec{a}$ in the plane as $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ for any two fixed noncolinear vectors $\vec{e}_1$, $\vec{e}_2$ (called *basis vectors*). In terms of the components (called *coordinates*), we have

$$\vec{a} + \vec{b} = (a_1\vec{e}_1 + a_2\vec{e}_2) + (b_1\vec{e}_1 + b_2\vec{e}_2) = (a_1 + b_1)\vec{e}_1 + (a_2 + b_2)\vec{e}_2 \tag{1}$$

(see fig.1) and

$$c\vec{a} = c(a_1\vec{e}_1 + a_2\vec{e}_2) = (ca_1)\vec{e}_1 + (ca_2)\vec{e}_2. \tag{2}$$

Thus this gives a correspondence between vectors and ordered pairs $(x_1, x_2) \in \mathbb{R}^2$ with the operations of addition and multiplication by numbers as defined below. The concept that relates the two is the *coordinate plane* (see fig.1). The same goes for 3-dimensional vectors, so one generalizes these concepts by introducing $\mathbb{R}^n$.

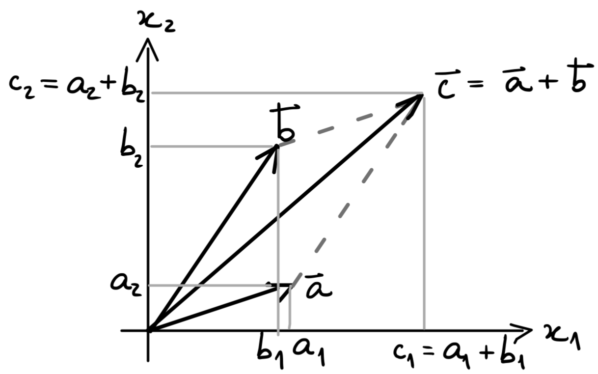

Figure 1. Vector addition in coordinates.

$\mathbb{R}^n$

We use notation $\mathbb{R}^n$ for the set of all points in n-dimensional space. Such a point is described by an n-tuple of numbers (coordinates) $x_1, x_2, \ldots, x_n$. We will write them as a column of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

By extending the ideas from the previous section, we will also refer to points of $\mathbb{R}^n$ as vectors (starting at the origin and ending at this point).

We have two natural operations on $\mathbb{R}^n$: addition of vectors and multiplication by numbers:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \qquad c \in \mathbb{R}$$

These operations satisfy obvious associativity, commutativity, and distributivity properties.
Note that there is no multiplication of vectors — only multiplication of a vector by a number.

Matrices $\mathbb{M}[m, n]$

A matrix of order $[m, n]$ is a generalization of the above concept: an array with m lines and n columns of the form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Sum of matrices of the same order and multiplication by a number are defined in analogy to the operations on $\mathbb{R}^n$ defined above.

The product of a $[m, n]$ matrix $A = [a_{ij}]$ by a $[n, p]$ matrix $B = [b_{ij}]$ is the $[m, p]$ matrix $C = AB = [\sum_{j=1}^{n} a_{ij} b_{jk}]$.

Linear functions

We will call a function $f : \mathbb{R} \to \mathbb{R}$ *linear* if $f(x+y) = f(x) + f(y), \forall x, y \in \mathbb{R}$ and $f(cx) = cf(x), \forall c, x \in \mathbb{R}$. We can generalize this concept by substituting the domain or contradomain by a space of vectors. For example, for a vector-valued function of vectors, these conditions would be

$$\vec{f}(\vec{x} + \vec{y}) = \vec{f}(\vec{x}) + \vec{f}(\vec{y}),$$
$$\vec{f}(c\vec{x}) = c\vec{f}(\vec{x}),$$

for all vectors $\vec{x}, \vec{y}$ and all $c \in \mathbb{R}$. This function could represent, for example, the velocity field of a fluid.

Homework

In the class, we saw that $\mathbb{R}^n$ is a good way to think of vectors. In these problems, you will understand how these relate to matrices and linear maps.

1. *Matrices are vectors*
   - (a) Write down the expression for $A + B$ and cA, where A and B are $[2, 2]$ matrices and $c \in \mathbb{R}$.
   - (b) Find 4 matrices e_1, e_2, e_3, e_4 such that every $[2, 2]$ matrix A can be written as $A = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4$.
   - (c) Write the operations of part (a) in terms of the coordinates (a_1, a_2, a_3, a_4).

2. *Linear functions are vectors*
   - (a) Show that, if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are linear functions, then $f + g : \mathbb{R} \to \mathbb{R}$, where $(f + g)(x) = f(x) + g(x)$, is a linear function. Then show that, for any $c \in \mathbb{R}$, the function $cf : \mathbb{R} \to \mathbb{R}$ defined by $(cf)(x) = cf(x)$ is linear.
   - (b) Show that $f(x) = ax$ for some $a \in \mathbb{R}$.
   - (c) Find a function $e_1 : \mathbb{R} \to \mathbb{R}$ such that any linear function $f : \mathbb{R} \to \mathbb{R}$ can be written as $f = f_1 e_1$ for some $f_1 \in \mathbb{R}$.

3. *Linear functions are matrices*

(a) Consider the equation $\vec{y} = f(\vec{x})$, where f is a linear function which takes 3-dimensional vectors as its argument and gives 2-dimensional vectors as its image. Now use bases to write $\vec{x} = x_1\vec{d}_1 + x_2\vec{d}_2 + x_3\vec{d}_3$ and $\vec{y} = y_1\vec{e}_1 + y_2\vec{e}_2$. Show that y_1, y_2 are linear functions of (x_1, x_2, x_3).

(b) Use the ideas from part (a) and from part (b) of the previous problem to show that there are real numbers a_{11}, a_{12}, a_{13} such that $y_1(x_1, x_2, x_3) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$. Find analogous a_{21}, a_{22}, a_{23} for $y_2(x_1, x_2, x_3)$.

(c) Write $\vec{y} = f(\vec{x})$ as a matrix equation.

(d) Show that, under the correspondence found between linear functions and matrices, sums of linear functions and product by a number (see previous problem) become sum of matrices and product of a matrix by a number.

4. *An interpretation of matrix product*

(a) Denote the space of two-dimensional vectors by $\mathbb{E}^2$. Consider two linear functions $f, g : \mathbb{E}^2 \to \mathbb{E}^2$. Show that $f \circ g : \mathbb{E}^2 \to \mathbb{E}^2$, defined by $(f \circ g)(\vec{x}) = f(g(\vec{x}))$, is linear.

(b) Using basis vectors $\vec{e}_1, \vec{e}_2$, we can write the equations $\vec{y} = f(\vec{x})$, $\vec{y} = g(\vec{x})$ and $\vec{y} = (f \circ g)(\vec{x})$ in matrix form, as in the previous problem. Let F and G denote the $[2, 2]$ matrices corresponding to the functions f, g, respectively. Show that the matrix corresponding to $(f \circ g)$ is the matrix product FG.