PARTIALLY ORDERED SETS

MARCH 22, 2020

A *partially ordered set* is a set S together with an order relation $<$ which has most of the usual properties of the order, namely:

1. Transitivity: if $a < b$ and $b < c$ then $a < c$

2. If $a < b$, then it is impossible that $a = b$ or $b < a$.

However, we do not require that any two elements are comparable: it might be that we have two elements $a \neq b$ such that neither $a < b$ nor $b < a$ hold.

There are many examples of such a relation, for example:

- Set of all figures in the plane, with relation being inclusion
- Set of all intervals on the plane, with relation being "interval I is to the left of interval J"
- Set of all people, with relation being "A is ancestor of B"
- Set of all words (say in English), with the realtion being "word A contains word B as its beginning" (e.g., "pen"<"pencil").

Problems

1. Given a sequence of 5 different integers, show that one can always selct three of them that form either increasing or decreasing subsequence

2. We have a number of boxes of different sizes and shapes. Some of boxes can fit inside others; however, it is also possible to have two boxes neither of which can fit inside the other. It is known that the largest "chain" of boxes that can be nested inside each other has depth 3: you can have three boxes such that A fits inside B which fits inside C, but you can not have 4 boxes such that each fits inside the next. Show that then, one can put the boxes in 3 piles so that in each pile no box can fit inside another (from the same pile).

3. Given a sequence of 10 diffferent numbers, show that one can select a subsequence of length 4 which is either increasing or decreasing.

[Hint: this is the same problem as the previous one: only relation "fits inside" replaced by $a < b$ and a, b are in correct order in the sequence.]

How long a sequence you need to take to guarantee existence of a monotonic subsequence of length 5?

4. Given 50 segments on the real line, show that one of the two properties must hold:

(a) it is possible to find 8 segments so that any two of them have a common point, or

(b) it is possible to find 8 segments so that no two of them have a common point.

[Hint: again, it is about a partial order on the set of segments...]