MATH CLUB: POLYNOMIALS AND VIETA FORMULAS

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Vieta formulas

Suppose that we have a polynomial of degree n with leading coefficient 1 which has been completely factored:

$$p(x) = x^n + a_1x^{n-1} + \dots + a_n = (x - x_1)\dots(x - x_n)$$

(thus, the roots of $p(x)$ are $x_1, \dots, x_n$).

Then one can express the coefficients $a_1, \dots, a_n$ in terms of roots $x_1, \dots, x_n$:

$$\begin{aligned}
&a_1 = -(x_1 + x_2 + \dots + x_n), \\
&a_2 = x_1x_2 + \dots \qquad \text{(sum of products of all distinct pairs of roots)} \\
&a_3 = -x_1x_2x_3 + \dots \qquad \text{(sum of products of all distinct triples of roots)} \\
&\quad \dots \\
&a_n = (-1)^n x_1 \dots x_n
\end{aligned}$$

These are called *Vieta formulas*. For $n = 2$, they become the usual formulas for quadratic equation: if $p(x) = x^2 + px + q = (x - x_1)(x - x_2)$, then $p = -(x_1 + x_2)$, $q = x_1x_2$.

Problems

1. Let x_1, x_2 be roots of the equation $x^2 + 13x - 7 = 0$. Find
   (a) $x_1 + x_2$
   (b) $\frac{1}{x_1} + \frac{1}{x_2}$
   (c) $x_1^2 + x_2^2$
   (d) $x_1^3 + x_2^3$

2. (2003 AMC 10A #18) What is the sum of the reciprocals of the roots of the equation

$$\tfrac{2003}{2004}x + 1 + \frac{1}{x} = 0$$

3. (2005 AMC 10B #16) The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of m, n, p is zero. What is the value of n/p?

4. (2006 AMC 10B #14) Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

5. (2001 AMC 12 #19) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has the property that the mean of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The y-intercept of the graph of $y = P(x)$ is 2. What is b?

6. One of the roots of equation $x^3 - 6x^2 + ax - 6 = 0$ is equal to 3. Find the other two roots.

7. Solve the system of equations:

$$\begin{gathered}
x + y + z = 6 \\
\tfrac{1}{x} + \tfrac{1}{y} + \tfrac{1}{z} = \tfrac{11}{6} \\
xy + xz + yz = 11
\end{gathered}$$

8. (1983 AIME) What is the product of the real roots of the equation

$$x^2 + 18x + 30 = \sqrt{x^2 + 18x + 45}$$

9. (1984 USAMO) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.