MATH CLUB: POLYNOMIALS AND ROOTS

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Some useful facts about polynomials

- **Long division**: given polynomials $f(x)$, $g(x)$ (with degree of $g(x)$ at least 1), one can uniquely write $f(x)$ in the form
$$f(x) = q(x)g(x) + r(x), \quad \deg r(x) < \deg g(x)$$
Polynomials $q(x)$, $r(x)$ are called *quotient* and *remainder* resepctively.
- **Bezout theorem**: when a polynomial $P(x)$ is divided by $(x-c)$, the remainder is $P(c)$. In particular, $P(x)$ is divisible by $(x-c)$ if and only if c is a root, i.e. $P(c) = 0$.
Moreover, if $P(x)$ has integer coefficients and c is an integer root, then $P(x)$ is divisible by $(x-c)$ and the quotient has integer coefficients.

Problems

1. Find the remainder when $x^{13} + 1$ is divided by $x - 1$

2. The polynomial $P(x)$ hasd remiander 99 when divided by $x - 19$ and remainder 19 when divided by $x - 99$. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?

3. Let $P(x)$ be a polynomial with integer coefficients and let a, b be integers, $a \neq b$. Prove that then $P(a) - P(b)$ is divisible by $(a - b)$.

4. Is it possible to find a polynomial with integer coefficients such that $P(7) = 11$ and $P(11) = 13$?

5. Prove that $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ if and only if n is not a multiple of 3.

6. Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.

7. Is it true that if the polynomial $P(x)$ is such that $P(n)$ is an integer for any integer n, then $P(x)$ has integer coefficients?

8. Construct a quadratic polynomial $f(x)$ such that $f(-1) = 1$, $f(0) = 0$, $f(2) = 4$.

9. A ship is traveling at constant speed keeping the same course without turning. The captain is measuring the distance to remote lighthouse every hour.
At noon, the distance was 10 miles.
At 3 pm, the distance was $10\sqrt{2}$ miles
At 5pm, the distance was $10\sqrt{10}$ miles.
What will the distance be at midnight?

***10.** Does there exist a polynomial with integer coefficients $P(x)$ such that for every integer n, $P(n)$ is a prime number?