MATH CLUB: COLORING AND CUTTING

JAN 5, 2020

1. Is it possible to cut a 10×10 board into 1×4 pieces? (All cuts must follow the grid lines.)

2. On every square of a 9×9 board there is an ant. At a signal, each ant moves to one of the squares diagonally adjacent to his. As a result, some squares will have more than one ant, and some will be empty.

What is the smallest possible number of empty squares?

3. As you know, in chess a knight moves one square to one side and two squares in perpendicualr direction.

Is it possible for the knight to start in the corner of a 7×9 board and visit each square of this board exactly once before returning to the starting position?

What about 4×8 board?

4. On a square ruled sheet of paper Sasha marked 2020 squares. Prove that it is possible to select 505 of them so that the selected squares have no common points (not even corners).

5. (a) Is it possible to cut a $6 \times 6 \times 6$ cube into $1 \times 2 \times 4$ pieces?
(b) Is it possible to put a $3 \times 3 \times 3$ cube with center piece removed into $1 \times 1 \times 3$ pieces?

6. Michael took a 29×29 square ruled sheet of paper and cut out 99 2×2 squares out of it (all cuts follow the grid lines). Show that the remaining part contains at least one more 2×2 square.

7. The streets of city X form a triangular grid so that each city block is an equilateral triangle. Traffci laws require that at each intersection, you can either go stright or turn 120° (left or right).

Two cars start from the same intersection in the same direction, but later take different routes. Can it be that they meet again in the middle of some street going in opposite directions?

8. The plane is colored using 3 colors: red, green, and blue. Prove that it is possible to find 2 points at distance 1 which have the same color.

***9.** Sasha took a 100×100 board and placed in some of the squares a stone (no more than one). Let's call a square x *nice* if the total number of stones in adjacent squares is even. (The square x itself doesn't count; "adjacent" means "'having a common side".)

Prove that it is impossible that exactly one square of the board be nice.

Hint to this problem is on reverse.

Hint to problem 9:
prove that for each square x, one can find a subset S of the squares with the following properties:

- S contains x
- S contains even number of squares
- Each square of the board has even number of adjacent squares from S