

PIGEONHOLE PRINCIPLE

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THE PIGEONHOLE PRINCIPLE

If you put n items in m boxes, with $n > m$, then at least one box will have more than one item.

Generalization

If $n > km$ objects are put in m boxes, then at least one box will have more than k objects.

PROBLEMS

1. Let S be a square region of the plane of side length 2. Prove that if 9 points are placed in S then three of them form a triangle of area at most $\frac{1}{2}$.
2. Prove that there exist integers a, b, c not all zero and each of absolute value less than one million, such that
$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$
3. Show that there is a positive term of the Fibonacci sequence which is divisible by 1000.
4. A sequence of m positive integers contains exactly n distinct values. Prove that if $2^n \leq m$ then there exists a block of consecutive terms whose product is a perfect square.
5. Let $1 = x_1 < x_2 < x_3, \dots$ be an increasing sequence of integers such that, for each n , $x_{n+1} \leq 2x_n$. Prove that every positive integer k can be represented as $x_i - x_j$ for some i, j .
6. Inside the unit square lie several circles, the sum of whose circumferences is at least 10. Prove that some line intersects four circles.
7. Let there be given nine lattice points in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.
8. Let n be an odd number greater than 1 and let A be an $n \times n$ symmetric matrix such that each row and column of A consists of some permutation of the integers $1, \dots, n$. Show that each one of the integers $1, \dots, n$ must appear in the main diagonal of A .