FORMAL POWER SERIES AND GENERATING FUNCTIONS

NOVEMBER 10, 2019

Formal power series

Given a sequence $a_0, a_1, a_2, ...$, its (formal) generating function is

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Generating functions can be added, subtracted and multiplied just like polynomials (collect like powers together). The *radius of convergence* of a power series is the largest number $R > 0$ such that, for all $0 \leq r < R$, $\sum_{n=0}^{\infty} |a_n| r^n$ converges. Within its radius of convergence, a power series may be differentiated and integrated like a polynomial (term-by-term), and they define a class of functions called *analytic functions*.

We have $f(0) = a_0$ and $f'(0) = a_1$. The nth derivative is $f^{(n)}(0) = n! a_n$. Hence, within the radius of convergece, the coefficients may be recovered from the power series.

Problems

1. Let $F(x) = F_0 + F_1 x + F_2 x^2 + \dots$ be the generating series for the Fibonacci numbers: $F_0 = 0, F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$. Prove that

$$F(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \Big(\frac{1}{1 - \Phi x} - \frac{1}{1 - \bar{\Phi} x} \Big)$$

where

$$\Phi = \frac{1 + \sqrt{5}}{2}, \qquad \bar{\Phi} = \frac{1 - \sqrt{5}}{2}$$

and use it to find an explicit formula for F_n.

2. Recall the *Catalan numbers* satisfy the recurrence $C_0 = 1$, $C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_n C_0$. Prove that

$$c(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

3. An *exact covering system of congruences* is a collection of arithmetic progressions, $0 \leq a_i < m_i$,

$$a_i + m_i k, \qquad k \in \mathbb{Z}$$

that are disjoint, and whose union is the integers. For instance, every integer satisfies exactly one of the congruences 0 mod 2, 1 mod 4, 3 mod 12, 7 mod 12, 11 mod 24, 23 mod 24.

a. Prove the generating function identity

$$\frac{1}{1 - x} = \sum_i \frac{x^{a_i}}{1 - x^{m_i}}.$$

b. Prove that every exact covering system of congruences with more than one congruence has a repeated step (e.g. 12 and 24 above).

4. Prove that $\sum_{m=0}^{\infty} \binom{2m}{m} \left(\frac{x}{2}\right)^{2m} = \frac{1}{\sqrt{1-x^2}}$.

5. Let $C(n)$ denote the number of ways of making n cents using pennies, nickels, dimes and quarters. Let $C(0) = 1$. Prove that

$$c(x) = \sum_{n=0}^{\infty} C(n) x^n = \frac{1}{1 - x} \frac{1}{1 - x^5} \frac{1}{1 - x^{10}} \frac{1}{1 - x^{25}}.$$

6. Let $P(n)$ be the number of partitions of n, that is, the number of ways of writing n as the sum of 0 or more positive integers. Thus $P(0) = 1, P(1) = 1$, $P(2) = 2$, $P(3) = 3$, $P(4) = 5$, etc. since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Prove that

$$\sum_{n=0}^{\infty} P(n)x^n = \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots .$$

7. A *permutation* σ of n letters is a one-to-one map of the numbers $\{1, 2, ..., n\}$. Permutations can be thought of as orderings of a deck of n cards. A *cycle* in a permutation is a sequence $a_1, a_2, ..., a_k$ such that $\sigma(a_1) = a_2, \sigma(a_2) = a_3, ..., \sigma(a_k) = a_1$. Every permutation can be split into disjoint cycles.

Given a set of numbers $b_1, b_2, b_3, ..., b_k$ with $b_1 + 2b_2 + 3b_3 + \cdots + kb_k = n$, let $c(b)$ denote the number of permutations of n numbers with b_1 cycles of length 1, b_2 cycles of length 2, ..., b_k cycles of length k. Let

$$C(x_1, x_2, ...; t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{b_1+2b_2+...+nb_n=n} c(b)x_1^{b_1}...x_n^{b_n}.$$

This is the 'grand' cycle index generating function.

a. Prove that

$$c(b) = \frac{n!}{\prod_{j\geq 1}(b_j!)j^{b_j}}.$$

b. Using the above or otherwise, prove

$$C(x_1, x_2, ...; t) = \exp\left(\sum_{j\geq 1} \frac{x_j t^j}{j}\right).$$