CATALAN NUMBERS AND MORE

NOVEMBER 3, 2019

Binomial coefficients identities

We use the notation

$$\binom{n}{k} = {}_nC_k = \frac{n!}{k!(n-k)!}$$

As you know, the binomial formula gives

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

1. Prove the following identities

(a)

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

(b)

$$\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0$$

(c)

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$$

2. Prove

$$\binom{n+m}{k} = \binom{m}{0}\binom{n}{k} + \binom{m}{1}\binom{n}{k-1} + \cdots + \binom{m}{k}\binom{n}{0}$$

Hint: interpret both sides as counting something, or as coefficients of some monomial...

Catalan numbers

Consider the sequence of numbers defined by

$$\begin{aligned} c_0 &= 1 \\ c_1 &= c_0 c_0 = 1 \\ c_2 &= c_1 c_0 + c_0 c_1 = 2 \\ &\dots \end{aligned}$$

$$c_{k+1} = c_0 c_k + c_1 c_{k-1} + \cdots + c_k c_0 = \sum_{i=0}^{k} c_i c_{k-i}$$

These numbers are called *Catalan numbers* and appear in many places.

3. Compute first 6 Catalan numbers, up to c_6

4. Consider expression

$$x_1 * x_2 * \cdots * x_{n+1}$$

where $*$ is some binary non-associative operation.

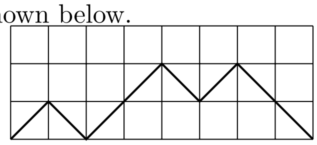
In order to make sense of this expression, we need to put in parentheses to indicate the order of operations. For example, for $n = 2$, there are two ways to do it: $(x_1 * x_2) * x_3$ and $x_1 * (x_2 * x_3)$.

(a) How many ways there are to put parentheses in product of 4 variables $x_1 * x_2 * x_3 * x_4$?

(b) Prove that there are exactly c_n ways to put parentheses in product of $n + 1$ variables [Hint: consider the operation performed last]

5. Prove that for a convex n-gon, there are exactly c_{n-2} ways to draw non-intersecting diagonals which would cut it into triangles.

6. A *Dyck path* is a polyline in the real plane which consists of segments $(1,-1)$ and $(1,1)$ (i.e., moving diagonally: one unit to the right and one unit either up or down), starts at $(0,0)$ and ends at $(2n,0)$ and which never goes below the x-axis (but may touch it). An example of Dyck path with $n=4$ is shown below.



We will denote the number of all Dyck paths of length $2n$ by D_n.

(a) Show that the number of Dyck paths of length $2n$ which are strictly above the x-axis (except the endpoints) is D_{n-1}.

(b) Show that $D_n = c_n$, i.e. the number of Dyck paths of length $2n$ is the Catalan number c_n (Hint: consider the first time the path touches the x-axis; use this point to divide the path into two subpaths).

(c) Show that the number of Dyck paths is the same as number of sequences of length $2n$, consisting of n symbols $+$ and n symbols $-$ such that in any initial segment of it, there are at least as many $+$ as $-$.

***7.** In this problem, you will prove that the number of Dyck paths of lenght $2n$ (and thus, the Catalan number c_n) is equal to

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

To do it, complete each of the steps below.

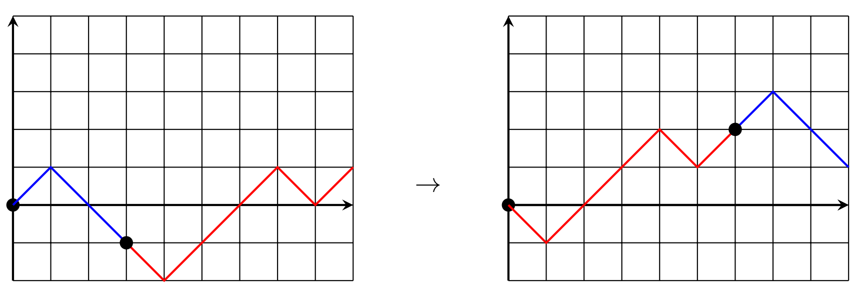
(a) Let S_n be the set of all paths consisting of n segments $(1,-1)$ (diagonally down) and $n+1$ segments $(1,1)$ (diagonally up), connecting points $(0,0)$ and $(2n+1,1)$. Show that the number of such paths is $\binom{2n+1}{n}$.

(b) Let us call such a path *positive* if it is strictly above x-axis (except point $(0,0)$). Show that the number of positive paths is the same as the number of Dyck paths of length $2n$ and thus is equal to the Catalan number c_n.

(c) Consider the following operation on S_n, which we will call *rotation by* k: given an integer k, $0 \le k \le 2n+1$,

- given a path p, divide into two pieces p_1 (with $0 \le x \le k$) and p_2 (with $k \le x \le 2n+1$)
- translate p_1 so that it starts at $(0,0)$
- translate p_2 so that it starts at the endpoint of p_1

The picture below illustrates this operation (for $k=3$)



Note that for $k=0$ and $k=2n+1$, the rotation does nothing: it leaves the path unchanged. Prove that for every path in S_n, there is exactly one rotation that makes this path positive. [Hint: consider the lowest point on the path.]

(d) Let us group paths in S_n together if one can be obtained from another by some rotation. Prove that then each group has exactly $2n+1$ paths in it, and that in each group, there is exactly one positive path.

(e) Prove that the number of positive paths (and thus, the Catalan number c_n) is given by

$$\frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$$

And now, for something completely different...

8. In this problem, we are talking about dice – fair six-sided dice which differ form the usual ones in that the numbers on the sides are not necessarily 1 through 6 but may be some other collection of numbers. We even allow the same number to be repeated more than once.

For a pair of such dice, A and B, we say that A beats B if when we roll them together, the number rolled on A will be larger than number rolled on B with probability $P > 0.5$.

Show that it is possible to construct 3 dice A, B, and C so that:

A beats B
B beats C
C beats A