INCLUSION–EXCLUSION FORMULA

OCTOBER 6, 2019

Combinatorics – simple problems

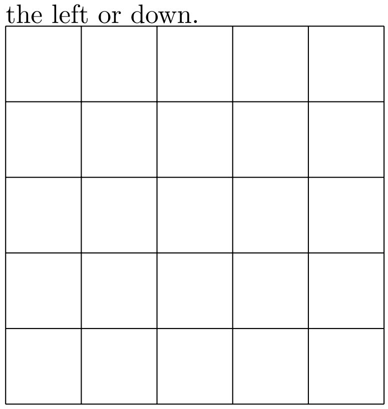
I expect most of you know how to solve these questions, but a reminder never hurts.

1. How many ways there are to seat 15 people in a room with 25 chairs?

2. There is a round table seating 8. How many ways there are for 8 people to choose their seats at the table? What if we do not distinguish between two seatings which only differ by rotating the table?

3. How many words one can get by permuting letters of the word "tiger"? of the word "rabbit"? of the word "common"? of the word "Mississippi"?

4. How many different paths are there on 5×5 chessboard connecting the lower left corner with the upper right corner? What about 6×6? The path should always be going to the right or up, never to the left or down.



5. A drunkard is walking along a road from the pub to his house, which is located 1 mile north of the pub. Every step he makes can be either to the north, taking him closer to home, or to the south, back to the pub – and it is completely random: every step with can be north of south, with equal chances. What is the probability that after 10 steps, he will end up

(a) 10 steps north from the starting position
(b) 10 steps south
(c) 8 steps north
(d) will come back to the starting position

Combinatorics – more advanced problems

6. You have 10 books which you want to put on 2 bookshelves. How many ways are there to do it (order on each bookshelf matters)?

7. How many monomials in variables x, y, z of total degree n are there? What if we had 4 variables? [Hint: it may be helpful if you write a monomial such as $x^3y^2z^8$ as $xxxyyzzzzzzzz$; thus, you just need to know when one kind of variable ends and another begins...]

8. How many ways are there to group $2n$ people in n pairs? (Pairs are not ordered: we do not care which pair was formed first and which was formed last. Also, inside each pair the two people are not ordered. All we care about is who is paired with whom.)

Combinatorics – inclusion-exclusion formula

It is well known that if we have two finite sets A, B, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(where $|A|$ is the number of elements in A).

One can generalise it to three sets:

$$\begin{aligned}|A \cup B \cup C| =& |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|\end{aligned}$$

It turns out that there is a similar formula for any number of sets:

$$\begin{aligned}|A_1 \cup A_2 \cup \dots \cup A_n| =& |A_1| + \dots + |A_n| \\ & - |A_1 \cap A_2| - \dots \qquad \text{(all possible pairwise intersections)} \\ & + |A_1 \cap A_2 \cap A_3| + \dots \qquad \text{(all possible triple intersections)} \\ & \dots\end{aligned}$$

This formula is called *inclusion-exclusion formula*

9. (a) Prove inclusion-exclusion formula for 3 sets.
(b) Prove inclusion-exclusion fromula for n sets. [Hint: if an element is exactly one of these sets, how many times will it be counted in the right-hand side of inclsuion-exclusion formula? what if it is in exactly 2 sets?...]

10. How many integers are there between 1 and 16500 which are not divisible by neither 3 nor 5? What about integers which are not divisible by either of 3, 5, 11?

11. How many ways there are to put 15 different toys in 4 boxes so that each box contains at least one toy? [Hint: let A_i, $i = 1 \dots 4$, be the set of those arrangements where box i is empty. Then $A_1 \cup A_2 \cup A_3 \cup A_4$ are those arrangements where at least one box is empty.]

12. Inclusion-exclusion formula has an analog where finite sets are replaced by figures in the plane and number of elements is replaced by area (we assume that all figures are nice enough so that notion of area makes sense and all areas are finite). State and prove this analog.

13. Inside a rectangle of area 1 we have 5 figures $F_1, \dots, F_5$, each of which has area 0.5. Prove the following:
(a) You can choose two of these figures so that their intersection has area at least 3/20.
(b) You can choose two of these figures so that their intersection has area at least 1/5.
(c) You can choose three of these figures so that their intersection has area at least 1/20.

14. A professor wrote 30 letters to his colleagues, and prepared 30 envelopes with their addresses. However, he was very much preoccupied with a theorem he was trying to prove and was not paying much attention to anything else, so he put the letters in the envelopes at random.

What is the probability that at least one letter was placed in the correct envelope?

[This is a famous problem which appears in many forms; it seems that it was first proposed in 1708 by a French mathematician Pierre Rémond de Montmort who called it "the problem of coincidences". A highly unexpected result is that if we replace 30 by n and ask how the probability behaves as n grows, it approaches a limit – and this limit is neither 0 nor 1. Many famous mathematicians worked on questions related to this problem, including Bernoulli, de Moivre, and Euler.]

***15.** (For those who have solved all previous problems)

We color all integer numbers between 1 and 1 000 000 using one of two colors, black and white. Initially all numbers are black. You are allowed to perform the following operation: chose a number n (between 1 and 1 000 000) and switch colors of n and all integers that have a common divisor > 1 with n.

Prove that using this operation, you can make all numbers white.