MATH GAMES CONTINUED

SEPTEMBER 29, 2019

General Info

We plan to participate in Harvard-MIT math tournament (HMMT) on Nov 9. Please discuss with your parents these plans and respond to the email I'll send shortly!

If you haven't yet done so, please bring \$50 (in cash or check payable to Alexander Kirillov) by next time – to cover pizza expenses.

Game Theory reminder

In all games we discussed last time, which are played by two players, each being allowed to make same kinds of moves, there are two kinds of positions:

- A **winning** position: a player who start in this position has a strategy that allows him to win regardless of his opponents moves
- A **losing** position: a player who starts in this position will lose, regardless of his moves, if his opponent uses the right strategy.

These positions can be found from the following rule:

For a given position, it there is a move from it to a losing position, then this posiiton itself is winning. Otherwise (i.e. if all moves from this position lead to winning positions), it is losing.

Wythoff's Nim

We discussed the following game, called Wythoff's Nim. The rules are below:

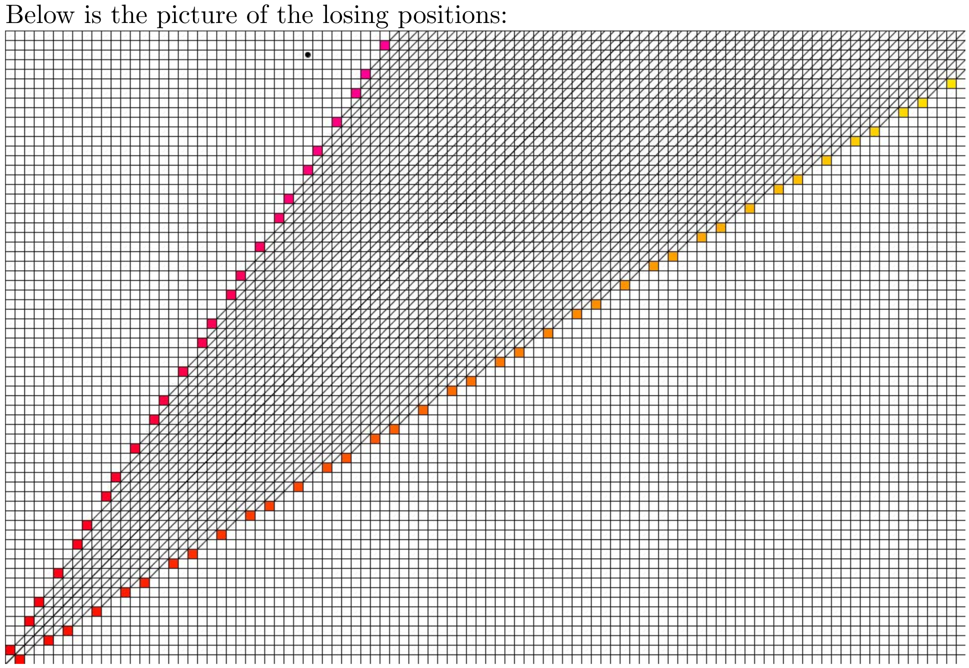
The game is played with two piles of stones, and the rules are as follows: at your turn, you can either remove any (nonzero) number of stones from one of the piles, or **the same** (nonzero) number of stones from both piles. As before, the player who has no moves loses.

To solve this game, it suffices to know which positions are losing. It is easy to see that positions

$$(0, 0), \quad (1, 2), \quad (3, 5), \dots$$

are losing.

Below is the picture of the losing positions:



Bottom left square is $(1, 1)$.

We have discussed in class the following properties:

1. If (x, y) is losing position, then so is (y, x); thus, it suffices to determine losing positions with $y > x$.

2. In each horizontal row, each vertical row, and each diagonal there is exactly one losing position.

Thus, if we denote by (x_n, y_n) the losing position on diagonal $y - x = n$, so that $y_n - x_n = n$, we see that each positive integer appears once as either x_k or as y_k.

Our goal is to prove the following result:

(1) $$x_n = [n\Phi], \quad y_n = [(n+1)\Phi]$$

where $[\alpha]$ is the integer part of α, and $\Phi = \dfrac{1+\sqrt{5}}{2}$ is the famous Golden Ratio.

To do that, we solve the series of problems.

1. Recall that by what was discussed above, the sequence of losing positions (x_n, y_n) has the following properties

- $x_1 = 1, y_1 = 2$
- $y_n = x_n + n$
- The sets $A = \{x_1, x_2, \dots\}$ and $B = \{y_1, y_2, \dots\}$ partition $\mathbb{N} = \{1, 2, 3 \dots\}$: every positive integer is either in A or in B, but not in both.

(a) Use these properties to construct first ten losing positions.

(b) Explain why these properties completely determine the sequence (x_n, y_n): there is only one sequence which has all of these properties.

2. For a positive real number α, denote

$$S_\alpha = \{[\alpha], [2\alpha], \dots\}$$

Let α, β be positive real numbers such that S_α, S_β partition $\mathbb{N}$: $S_\alpha \cup S_\beta = \mathbb{N}$ and S_α, S_β do not intersect.

(a) Show that both α, β are greater than 1.

(b) Show that if $1 < \alpha < 1.1$, then $\beta \geq 10$

(c) Show that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

[Hint: if we take all numbers $1, 2, \dots N$, for some very large N, how many of them are in S_α? in S_β?]

(d) Show that α, β must be irrational.

3. Converesley, show that if α, β are positive irrational numbers satisfying

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

then S_α, S_β partition $\mathbb{N}$.

4. Let α be such that $1 < \alpha < 2$ and

$$\frac{1}{\alpha} + \frac{1}{\alpha+1} = 1$$

Let $x_n = [n\alpha]$, $y_n = [n(\alpha+1)]$. Show that then the sequence (x_n, y_n) has all the properties listed in Problem 1 and thus is the sequence of losing positions for Wythoff's Nim.

5. Solve equation

$$\frac{1}{\alpha} + \frac{1}{\alpha+1} = 1$$