MATH CLUB: FIRST MEETING

SEPTEMBER 22, 2019

General Info

In the Problem Solving Club, we will be solving fun (but hard) problems, run math battles, and paricipate in math olympiads such as AMC, Harvard-MIT Math Tournament, and Math Kangaroo. We also give new material — some new theories, or new methods of problem solving — but we will not follow any standard syllabus.

Oh, and we will also be eating pizza!

The class will be taught by Alexander Kirillov ⟨`kirillov@schoolnova.org`⟩. If you have any questions or concerns, please feel free to email me!

To cover pizza expenses, I ask that each student bring $50 (in cash or check payable to Alexander Kirillov) by next time.

Game Theory

In all of the games we will be dealing with there are 2 players. Because these are mathematical games, one of the players always has a winning strategy. Your goal is to figure out who has the winning strategy and what is.

A winning strategy is an algorithm which guarantees a win for the corresponding player REGARDLESS of his opponents moves. It must be accompanied by a proof of why this algorithm is indeed a winning one. Argument "I tried it several times and it seems to work" is not enough.

Here are some ideas you can use for constructing such a strategy:

1. **Winning/Losing Position:** A winning position is a position such that if a player is in this position (before his move), he is guaranteed to win the game. The most important idea here is that if a player is in a certain position and regardless of which move he makes, his opponent will be in a winning position, then that certain position is a losing position. On the other hand, if a player is in a certain position, and can always make a move so that his opponent will be in a losing position, then that certain position is a winning position.
2. **Work Backwards:** Think about when the game ends. What must be the move(s) that happened right before the game ends? You can figure out some winning and losing positions, and from there work your way backwards to figure out if the starting position is winning or losing.
3. **Symmetry** A very simple strategy is to copy or almost copy your opponent's moves. Here is an example. Two players take turns placing identical coins on a round table. The coins cannot overlap. A player loses if they can't make a move. Who can guarantee to win?

   Solution: The first player should place a coin in the exact center of the table. Then she places every next coin in the position that is symmetric with respect to the center to the position of the last coin placed by her opponent.

Your task is analyzing each of the games below to find a winning strategy for one of the players. Each is played by two players.

Some simple games

1. (a) There are 25 matches on a table. During each turn, a player can take any number of matches between 1 and 4. The player that takes the last match wins.
   (b) Same game as above but it starts with 24 matches.
   (c) Same game again, only the initial number of matches is n.
2. Two players are moving the hour hand of the clock. Each player can move it forward (clockwise) either 2 or 3 hours. Initially the hand points to 1; the player who moves it to 12 wins.
3. At the start of the game, there is a number 60 written on the board. During each turn, a player can subtract from the number that is currently on the board one of its positive divisors. If the resulting number is a 0, the player loses.

4. On one square of an 8 by 8 chessboard there is a "lame tower" that can move either to the right or up by any number of squares.Two players take turns moving the tower. The player unable to move the tower loses. (Consider various initial positions of the tower.)

5. Given a convex n-gon, the players take turns drawing diagonals that do not intersect those diagonals that have already been drawn. The player unable to draw a diagonal loses.

6. There are 5 buckets arranged in a row. Every bucket contains a marble. On every move a player can select one of the buckets (except the rightmost bucket) and move all the marbles in the bucket to the neighboring bucket to the right. The player who cannot make a move loses.

7. Start with a rectangular chocolate bar which is 6×8 squares in size. A legal move is breaking a piece of chocolate along a single straight line bounded by the squares. For example, you can turn the original bar into a 6×2 piece and a 6×6 piece, and this latter piece can be turned into a 1×6 piece and a 5×6 piece. The player who can't make a move loses.

What about the general case (the starting bar is $m \times n$)?

More advanced games

8. Two people are playing a game, moving pieces on a plane. Player one controls a black piece ("wolf"); during his turn, he can move the wolf in any direction by not more than 1cm. Player two controls 100 white pieces ("sheep"); during his turn, he can move **one** of the sheep in any direction by not more than 1cm. Player one starts.

Is it true that no matter what the initial positions were, the wolf will be able to get at least one sheep?

9. The SOS game (from USA Math Olympiad, 1999).

The board consists of a row of n squares, initially empty. Players take turns selecting an empty square and writing either an S or an O in it. The player who first succeeds in completing SOS in consecutive squares wins the game. If the whole board gets filled up without an SOS appearing consecutively anywhere, the game is a draw.

(a) Suppose $n = 4$ and the first player puts an S in the first square. Show the second player can win.
(b) Show that if $n = 7$, the first player can win the game.
(c) Show that if $n = 2000$, the second player can win the game.
(d) Who wins the game if $n = 14$?

10. Game of Nim

Start with several piles of stones. A legal move consists of removing one or more stones from one of the piles. The player who can't make a move (i.e. when there are no stones left) loses.

The solution is obvious if we have only one pile, so we consider two or more

(a) Analyze the game when there are two piles of stones, m and n.
(b) Analyze the game with three piles of 17, 11, and 8 stones, respectively.
(c) Analyze the case of three plies, with k, m, n stones respectively.
Hint: define the *nim-sum* $m \oplus n$ of two numbers m, n as follows: write each number in binary and do the binary addition **without carrying**, e.g.

$$3 \oplus 5 = 011_2 \oplus 101_2 = 110_2 = 6$$

Show that this operation is associative and commutative, and watch the nim-sum of all the piles.
(d) Can you suggest a general solution??

11. Wythoff's Nim. Consider a variation of the game of Nim played with two piles of stones, and the rules are as follows: at your turn, you can either remove any (nonzero) number of stones from one of the piles, or **the same** (nonzero) number of stones from both piles. As before, the player who has no moves loses.

(a) Analyze the game with the number of stones 14 and 10.

(b) Show that $(1, 2)$ and $(3, 5)$ are losing positions. Can you find more losing positions? Can you find a pattern or general rule?
[This is a very hard problem. For starters, can you prove that for every n, there is exaclty one losing poisiton (x_n, y_n) such that $y_n - x_n = n$?]

AND NOW FOR SOMETHING COMPLETELY DIFFERENT...

12. Eleven millipedes are trying to climb the Glass Mountain. The first millipede has 20 legs, the second, 22 legs, and so on, up to the last one that has 40 legs.

The slopes of the mountain are slippery, so to climb it, a millipede must put special climbing shoes on at least half of its feet.

What is the smallest possible number of shoes they need to climb the mountain?

[Small print: initially, all millipedes are down at the foot of the moutnain. At the end, they all must be at the top together. Throwing shoes down the mountain is not allowed. The only way a millipede can carry the shoes up or down the mountain is on its feet — no more than one shoe per foot.]