A and G 1. Classwork 10.

Algebra.

Mathematical logic.

A speech of a person or a text written on paper contains sentences. This is the way in which we exchange information between us. The information in every sentence can be a true fact, false, or sometimes we just can't say whether it is true or false. For example, the sentence:
"The Earth is rotating around the Sun" is true.
"Paris is the capital of Germany" is false.
"Math is fun!" or "What time is it?" are the sentences we can't tell either they are true or false.
Can you determine whether sentences below are "true", "false", and for which ones we cannot tell?

a. "22 is an even number"
b. "44 is an odd number"
c. "1001 is a cool number!"

Let's define "a statement" as a sentence about which we can tell (sometimes after a difficult process of proving) either it is true or false. For example, our first sentence "The Earth is rotating around the Sun" was proven to be true after hundreds of years of discussions. The second sentence, "Paris is the capital of Germany", can be proven wrong after we will check it in the dictionary (assuming that we never took geography class). As for the third example, how we can tell is 1001 a cool number? What is "cool"? for whom? According to the definition of the statement, the sentence "22 is an even number" is a statement, and this is a true statement. "44 is an odd number" is also a statement, but the false one. "1001 is a cool number!" is not a statement at all.

1. Which of the following sentences are statements?
   a. When is the first day of school this year?
   b. The 4th of July is Independence Day.
   c. How beautiful is it!
   d. Washington, DC is the capital of the United States.
   e. The sum of five and three.
   f. Three times five is twenty-six.

2. Which of the following statements are true, and which are false?
   a. There are 31 days in each January.
   b. There are 28 days in each February.
   c. Sunday is followed by Tuesday.
   d. There are 7 days in each week.
   e. There are 7 letters in the word "table"
   f. The sum of all single-digit natural numbers is equal to 45.
   g. Every 3-digit natural number is greater than 100.

h. There exists the greatest 5-digit natural number.
i. There exists the greatest natural number.
j. There exists the smallest natural number.

Let's take a look at the statement "New York City is the capital of the United States". We, definitely, can say is it True or False. Of cause it's not true, we all know that the capital of the US is Washington, DC. So, we can say, "it is not true, that New York City is the capital of the US", or, in a little more usual language, "New York City isn't the capital of the US". The last statement is a true statement.

"New York City is the capital of the United States" False
"New York City is not the capital of the US" (negation) True

If a given statement is True, its negated version has to be False and vice versa. They can't be both True or both False. This rule of math logic is one of the oldest and is called "The law of the excluded middle".

3. Let's try to construct negation of the several statements.
   a. Number 111111111 is a prime number.
   b. There is nothing on the table.
   c. 0.5 and $\frac{1}{2}$ are not equal.
   d. The area of a rectangle is equal to the product of its length and width.
   e. The number given by $18 \cdot 946 + 456$ is divisible by 9.
   f. $45784 > 45784$
   g. $345 < 12345$
   h. All birds can fly.
   i. All marine animals are fish.
   j. Some students like math.
   k. All natural numbers are divisible by 3.
   l. Penguins live on the North Pole.
   m. Polar bears live on the South Pole.

4. Using the law of the excluded middle prove, that the negations of statements below were constructed incorrectly.

	Statement	Negation
1	All cats are gray.	All cats are not gray
2	Some berries are sweet.	Some berries are not sweet.
3	There are 30 days in some months.	There are no 30 days in some months.
4	All birds can fly.	There are no birds that can fly.

A categorical statement is a statement about the relationship between categories or classes of objects. It states whether one category is fully contained within the other, is partially contained (there is at least one member of the category) within the other, or is completely separate.
"Any (all) natural number is divisible by 3" is a categorical statement and it is a false statement. It is very easy to prove that it is false by giving a counterexample: 5 is a natural number and it is not divisible by 3. It means that not any natural number is divisible by 3, some of them are not and it is sufficient to present only one such number to prove that the statement is false.

“The sum of even numbers is an even number” is a categorical statement. The statement is about the category “sum of two arbitrary even numbers”, this category belongs to the set of even numbers. We can either prove it wrong by showing at least one example of the odd sum of two even numbers or prove it true by reasoning. It is not enough to show several examples to prove that this statement is true and, of course, in this case, there are no examples to prove it wrong.

Proof. Any even number can be represented as $2k$ (or $2n$) where $k, n \epsilon N$

$$2k + 2n = 2(k + n), \quad k, n \epsilon N$$

So, the sum is divisible by 2, or even number.
How the negation of the categorical statement can be done?
If the statement is about the whole category (all elements of the category) which forms a subset (we can use the set theory formalism here) of another category, the negated statement will show the existence of at least one element of the set, which doesn’t belong to the category.

“All birds can fly. “ – the statement is telling us that the whole category (all birds) belongs to another category, things that can fly. The negation of this statement should tell us that there is at least one element (one kind of birds) that can’t fly. We can formulate it as:
“Some birds can’t fly” or as “There are birds that can’t fly.”

$P =$ “All men are not bald” , $\neg P$ = “Some men are bald” and vice versa:
$P =$ “Some cats are gray”, $\neg P$ = “All cats are not gray”

We can create more complex statements, for example

I like math and physics. (I like math AND I like physics.)

In which case this statement will be a true statement?

1. I like math, but I don’t like physics.
2. I like physics, but I don’t like math.
3. I like both, math and physics.
4. I don’t like math, and I don’t like physics.

To negate the statement, we have to make a statement that is false whenever the original statement is true and vice versa. Can you make a negation of the statement above?
This statement is made by composing two other statements: “I like math” (statement A) and “I like physics” (statement B). Let’s create a Truth table for the composite statement **A** AND **B**

A	**B**	**A ∧ B (AND, ·)**
T	T	T
T	F	F
F	T	F
F	F	F

The negation of each of the statements A and B are “I don’t like math”, “I don’t like physics”. The truth table of **¬ A** (not A) OR **¬ B** (not B) is below. In this case, the whole statement is true if at least one of the statements is true (or both statements are true).

A	**B**	**A ∧ B (AND, ·)**	**¬ A**	**¬ B**	**¬ A∨ ¬ B (OR, +)**
T	T	T	F	F	F
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

From this truth table, we can see that the negation of the statement “I like math and physics” should look like “I do not like math OR I do not like physics” and is true whenever the original statement is false and vice versa. Here we used a logical operation OR.

The truth table for OR:
Statement P = ”I eat pears or apples after dinner”
Statement A = “I eat pears after dinner”, statement B = ”I eat apples after dinner”.

A	**B**	**A OR B (∨)**
T	T	T
T	F	T
F	T	T
F	F	F

In math logic, the statement “A or B” is true if A is true, B is true, both A and B are true”. Please, do not confuse this with the construction “either…or” which in common language usually means one out of two possibilities, not both as in: “I eat either pears or apples after dinner”.

Exercises:
1. Which of the following statements are categorical statements?
    a. Some types of plants and animals are listed in the list of endangered species.
    b. All planets of the Solar system are rotating around the Earth in the same direction.
    c. Some butterflies are yellow.
    d. There are 22 books on the shelf.
    e. Any natural number is greater than 0.

2. Find one counterexample to prove the following statements wrong:
    a. All natural numbers are greater than 1.
    b. Any number divisible by 5 ends with digit 5.
    c. All rivers of the United States flow into the Pacific Ocean.
    d. All marine animals are fish.
    e. All American cities lie south of 50 degrees latitude.

3. The mother told Mary, that she could play videogames if she will do her homework, also will do her room, and will do dishes after dinner. Will Mary play videogames if she
    a. Did her room?
    b. Did her room and dishes?
    c. Did her homework?
    d. Did her homework and dishes?
    e. Did all three assignments?
4. On the other day, the mother told Mary that she will play videogames if she will do her homework or will do dishes. Will Mary play the videogames if she
    a. Did her homework?
    b. Did the dishes?
    c. Did both assignments?
5. The following are some statements with proofs. Can you tell which proof is correct and which is wrong?
    a. All natural numbers are divisible by 7. Proof: for example: 14:7=2.
    b. Some proper fractions have denominator equals to 8. Proof: for example, the denominator of the fraction $\frac{3}{8}$ is 8.
    c. There are even numbers that are multiples of 3. Proof: for example, 36 is a multiple of 3, 36:3=12.
    d. Some nouns in English contain 5 letters. Proof: for example, “table”.
    e. All verbs in English start with the letter “w”. Proof: for example, “ (to) write”.
    f. There are books written by Joanne Rowling. Proof: for example, “*Harry Potter and the Philosopher's Stone*”.
6. Inhabitants of the city A always tell the truth, inhabitants of the city B always lie, and inhabitants of the city C tell truth or lie every other time. The fire department got a call: “We got fire here, come over as soon as you can”. “Where is the fire?” - asked the fireman. “In the city B”, - was the answer. Where should they go, if the fire in one of the cities is real?
7. Simplify the following expressions for valid variable values (it means that we are not dividing by 0 anywhere).

$\frac{1}{a^3}(-3aa^4)$ $\qquad$ $\frac{-3x^2\cdot(-xy)^3\cdot x^0\cdot y^0}{(x^2)^3\cdot(-3y)^2}$ $\qquad$ $\frac{(4bc^3)\cdot(-ac^2)^2\cdot(2a^2b^3c)^3}{(-2b^2c^2)^5\cdot(((-a)^2)^2)^2}$

$(-2b^3)^5\cdot\left(-\frac{1}{2b^3}\right)^3$ $\qquad$ $\frac{(m^2n)^3\cdot(mn^4)\cdot(-25m)^2}{(-5m^3n^2)^3\cdot(mn)^0}$ $\qquad$ $\frac{(zyx^2)^4\cdot(7y^2)^3\cdot(2x^2z)^2}{(14y^5z^3)^2\cdot(-((-x)^2)^2)^3}$

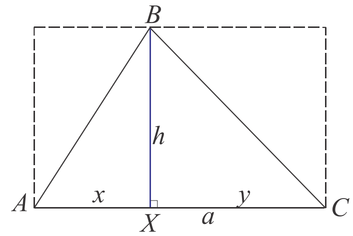
8. Jane and Mary are doing fall clean up in a backyard. Mary can do the job in 6 hours; together they can do it in 4 hours. How many hours does Jane need to clean up the backyard alone?
9. 5 hamsters eat 5 bags of hamster food in 5 days. How many days do 10 hamsters need to eat 10 bags of food?
10. This year twice as many students participated in the math competition than last year. By how many percent did the number of participating students increase?
11. This year two times less students missed the school than last year. By how many percent did the number of sick students decrease?

Geometry.

Area of a triangle. Area of a trapezoid and parallelogram.

The area of a triangle is equal to half of the product of its altitude and the base, corresponding to this altitude.

For the acute triangle, it is easy to see.

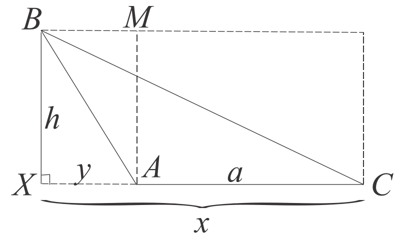


$S_{rec} = h \times a = x \times h + y \times h$

$S_{\Delta ABX} = \frac{1}{2} h \times x, \qquad S_{\Delta XBC} = \frac{1}{2} h \times y, \qquad S_{\Delta ABC} = S_{\Delta ABX} + S_{\Delta XBC}$

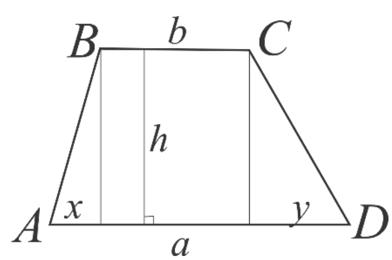
$$S_{\Delta ABC} = \frac{1}{2} h \times x + \frac{1}{2} h \times y = \frac{1}{2} h(x + y) = \frac{1}{2} h \times a$$

For an obtuse triangle, it is not so obvious for the altitude drawn from the acute angle vertex.



$$S_{\Delta ABC} = \frac{1}{2} h \times x - \frac{1}{2} h \times y = \frac{1}{2} h(x - y) = \frac{1}{2} h \times a$$

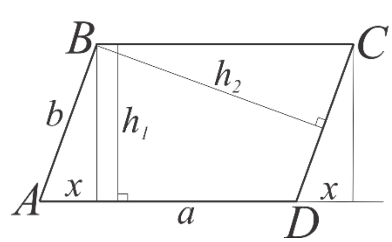
Area of the trapezoid can be calculated as



$$S = \frac{1}{2} hx + \frac{1}{2} hy + hb = \frac{1}{2} h(x + y) + hb = \frac{1}{2} h(a - b) + hb$$

$$= h\left(\frac{1}{2} a - \frac{1}{2} b + b\right) = \frac{1}{2} h(a + b)$$

And for parallelogram we can write:



$$S = h_1 a = h_2 b$$

Can you explain why?