# 1 Math 4 - Prerequisites for Math 5

## 1.1 Program

- Division with remainder. Divisibility tests by 2, 3, 4, 5, 6, 9. Divisors (factors), multiples.
- LCM, GCD. Finding by listing of all divisors.
- Prime and composite numbers. Prime factorization.
- Finding GCD and LCM using prime factorization.
- Fraction: addition and subtraction. Comparison.
- Multiplication and division of fractions. Finding a fraction of a number. Finding a number given its fraction.
- Speed, time, distance problems.
- Basic geometric concepts. Angles.
- Sum of angles of a triangle and a polygon.
- Quadrilaterals: parallelogram, rectangle, square, rhombus.
- Areas. Area of triangle, trapezoid, parallelogram.
- Negative numbers. Addition, subtraction, comparison. Absolute values. Multiplication and division of negative numbers.
- Distributivity. Opening the parentheses.
- Solving basic equations, including ones with negative numbers.

## 1.2 Problems

**1.** Find the following sums

$$1 + 2 + 3 + \cdots + 49$$

$$1 + 3 + 5 + \cdots + 49$$

**2.** Is the number 12345 divisible by 3? by 9? by 5? by 10?

**3.** If it is 7am now, what time of the day will it be in 27 hours? 127 hours? 11043 hours?

**4.** Find the LCM and GCD of 28 and 30.

**5.** A package of plastic forks contains 16 forks. A package of plastic knives contains 12 knives. What is the smallest number of packages of each kind you have to buy to get exactly the same number of forks as knives?

**6.** Find a prime factorization of 204.

**7.** Find LCM and GCD of 365 and 30.

**8.** Consider the product of all numbers from 1 to 25: $1 \times 2 \times \cdots \times 24 \times 25$. How many 3s will there be in the prime factorization for this number?

**9.** Compute $\frac{14}{7} + \frac{45}{11}$, $\frac{7}{10} - \frac{1}{2}$.

**10.** Compare $\frac{11}{6}$ and $\frac{7}{4}$.

**11.** Compute

$$\text{(a) } \frac{3}{14} \times \frac{7}{9} \qquad \text{(b) } \frac{12}{33} \times \frac{55}{56}$$

**12.** Compute

$$\text{(a) } \frac{3}{14} \div \frac{7}{9} \qquad \text{(b) } \frac{12}{33} \div \frac{55}{56}$$

**13.** Mrs. Weatherby baked 175 cookies for a party. The children ate $\frac{4}{7}$ of the cookies. The adults ate 48 cookies. How many cookies were left?

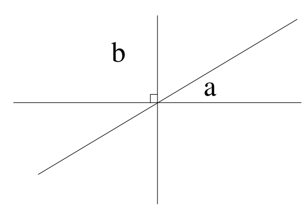
**14.** There are 4 short stories in the book. The first story is 12 pages long, which is $\frac{2}{3}$ of the second story. The third story is $\frac{5}{6}$ of the length of the first two stories together. How long is the fourth story, if four stories together occupy 64 pages in the book?

**15.** A boat has speed of 8 miles per hour (mph).

(a) Two towns, A and B, are on the shores of a lake. How long would it take the boat to go from A to B and back if the distance between the towns is 10 miles?

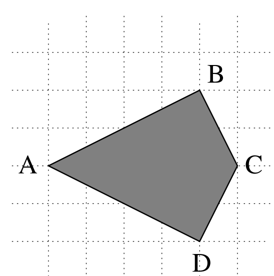
(b) Two other towns, C and D, also 10 miles apart, are on a river: C is upstream, D is downstream. The river flows at the speed of 2 mph. How long will it take the boat to go from C to D? from D to C?

**16.** In the figure on the right, $\angle a = 30°$ and $\angle b$ is the right angle. Can you find the sizes of all other angles in the figure?

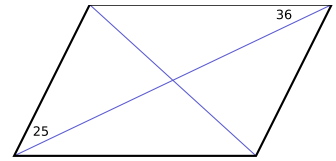



**17.** Find the angle between the two clock hands at 12:20.

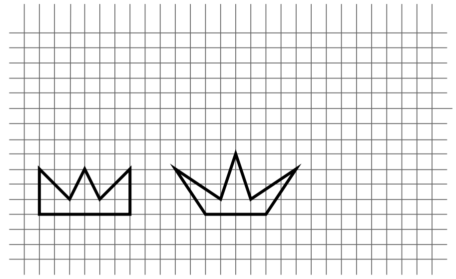
**18.** A hardware store is selling floor tiles of the shape shown to the right (for your convenience, it is shown on quad ruled paper). They are symmetric around line AC and have angles $\angle A = 52°$, $\angle B = \angle D = 90°$, $\angle C = 128°$. Do you think it is possible to tile the floor with these tiles without gaps? If so, can you show how?



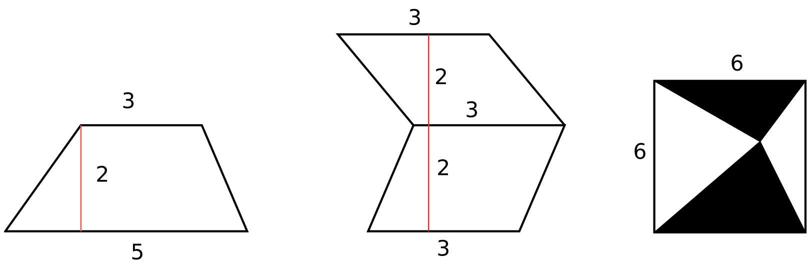
**19.** Two of the angles between the diagonal and the side in parallelogram $ABCD$ are marked. Can you find all other angles – including the angles between sides and diagonals, and the angles between diagonals?




**20.** Find the area of the figures show to the right.



**21.** Compute the area of the figures below. The picture is not to scale, so do not try measuring the lengths - use the numbers given. In the last one, find the area of the shaded part.




**22.** Compute

$$(-7) + (-9) = \qquad 3 + (-6) + (-7) = \qquad (-3) + 5 + (-7) =$$

**23.** Solve the equations

$$x - 12 = -10 \qquad z + (-6) = -15 \qquad |x| = 3 \qquad |5 + x| = 3$$

**24.** Compute the following expressions:

$$(-6) \div (-2) + 3 \qquad (-2) \div (-3) \qquad (-4) \times (-7) \div 9$$

**25.** Solve the following equations

$$(-2) \times x = -7 \qquad (-3) \times x + 2 = x - 18$$

**26.** Simplify the following expressions

$$2(x + y) - 2(x - y) \qquad 1 - 2(1 - 2(1 - 2x))$$

**27.** Solve the following equations.

$$5(x-2) = 25 \qquad 4x = 2x + 8 \qquad (-2x) + 3 - (-5x) - (-7) = -(-1)$$

## 2 Math 5 — Prerequisites for Math 6

### 2.1 Program

- Algebraic expressions. Commutativity, associativity, distributivity.
- Equations. Solving word problems with equations.
- Powers of 2.
- Binary numbers.
- Powers. Negative powers. Scientific notation.
- $a^2 - b^2 = (a-b)(a+b)$
- Squre roots.
- Pythagorean theorem.
- Basic probability theory: addition rule, complement rule, product rule.
- Choosing with and without repetition. Permutations.
- Geometry: parallel lines and angles (alternate interior, alternate exterior, corresponding).
- Parallelogram, various definitions, properties.
- Congruence tests for triangles (SAS, ASA, SSS).
- Isosceles triangle. Median, bisector, height.
- Trapezoid. Its midline. Area.

### 2.2 Problems

**1.** Rewrite each of the expressions below in the simplest possible form, by collecting the like terms if possible.

$$\begin{array}{lll} 2x+7+5x+2+3x & 3x+9+5xy+2xy+3 & 3(2x-1)+x \\ 2a(a-2)-a(a-1) & (2x-1)(x+1) & \end{array}$$

**2.** An apple cost 9 cents, and an orange 15 cents. Elena bought some apples and oranges, 20 fruit in all, and paid $2.64. How many apples and how many oranges did she buy?

**3.** A boy had a bag of apples. He gave 1/2 of them to his parents, 1/5 to his brother, 1/4 to his sister and the last apple he ate himself. How many apples did he originally have?

**4.** Simplify the following expressions

(a) $x+4(1-x)$ (b) $2+5x-4(3-x)$ (c) $5(x-1)-3(2x+1)$

**5.** If you take half my age and add 7, you get my age 13 years ago. How old am I?

**6.** Two secretaries, Barbara and Mary, need to type a 100 page document. Barbara can type it in 4 hours; Mary types slower, so it would take her 5 hours to do this. How fast can they type it together if they divide the work between two of them in the most efficient way?

**7.** Find the sum $1 + 2 + 4 + \cdots + 2^n$ (the answer, of course, will depend on $n$). [Hint: first try computing it for several small values of $n$: find $1 + 2$, then $1 + 2 + 4$, then $1 + 2 + 4 + 8$. See if you can notice a pattern. After this, try formulating a general rule.]

**8.** Convert the decimal numbers to binary:

9, 12, 24, 38, 45

**9.** Convert the folowing binary numbers to decimal:

101, 1001, 10110, 11011, 10101

**10.** Compute $110101_b + 111011_b$ without converting numbers to decimal form.

**11.** Simplify the following expression:

$$\frac{(x^2y^2) \cdot x^3}{x^2y^5}$$

**12.** Let $a = 2 \cdot 10^8$, $b = 10^5$. Compute $a^2 \cdot b$, $\frac{a}{b}$, $a^2 \div b^3$.

**13.** If $a = 2^{-13}3^9$, $b = 2^{11}3^{-7}$, what is the value of $ab$? of $a/b$?

**14.** Write the following numbers using scientific notation.

(a) the distance from Earth to Pluto is $\approx$ 7,527,000,000 km;

(b) the distance from Earth to the star Sirius is $\approx$ 81,900,000,000,000 km;

**15.** Factor the following number into primes: $99^2 - 9^2$. [Hint: you do not have to compute this number.]

**16.** Find the following square roots. If you can not find the number exactly, at least say between which two whole numbers the answer is, e.g., between 5 and 6.

(a) $\sqrt{81}$

(b) $\sqrt{10,000}$

(c) $\sqrt{10^8}$

**17.** If, in a right triangle, one leg has length 1 and the hypotenuse has length 2, what is the other leg?

**18.** Simplify: $(\sqrt{17})^2$, $(\sqrt{13})^4$, $(\sqrt{11})^3$, $\sqrt{2^4\,3^6}$, $\sqrt{2^4\,3^5}$.

**19.** We roll two dice. What is the probability of getting sum of two numbers equal to 4?

**20.** If we toss a coin 5 times, what is the probability that **at least one** will be heads?

**21.** A license plate consists of 3 letters, followed by three digits. How many possible license plates are there?

**22.** In one kind of lottery, they put balls with numbers 1 through 100 in a bag and then draw six balls at random (drawn ball is put aside and not returned to the bag). To win the lottery, one needs to guess all six numbers in correct order. What is the probability of this?

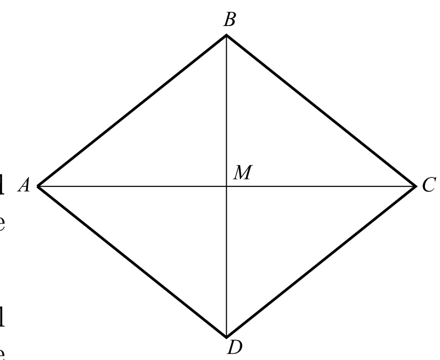
**23.** How many ways are there to seat 15 students in a classroom which has 15 chairs? If the room has 25 chairs?

**24.** In a meeting of 25 people, each person must shake hands with each other. How many handshakes will there be altogether?

**25.** At a certain meeting of 25 people, they decide to select a committee, which would have a chairman and 2 members. How many ways are there to do it?

**26.** Show that in a parallelogram, diagonally opposite angles are equal $\angle A = \angle C$, $\angle B = \angle D$

**27.** Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called rhombus). Let $M$ be the intersection point of $AC$ and $BD$.



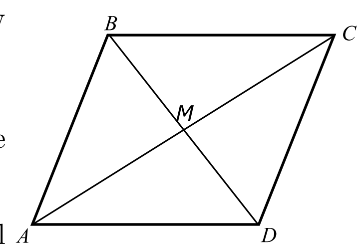

(a) Show that $\triangle ABC \cong \triangle ADC$

(b) Show that $\triangle AMB \cong \triangle AMD$

(c) Show that the diagonals are perpendicular and that the point $M$ is the midpoint of each of the diagonals.

[Hint: after doing each part, mark on the figure all the information you have found — which angles are equal, which line segments are equal, etc: you may need this information for the following parts.]

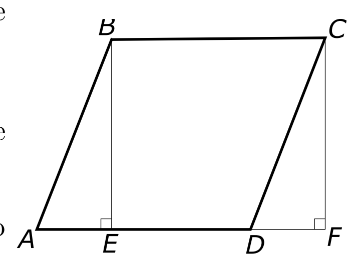
**28.** Let $ABCD$ be a quadrilateral such that sides $AB$ and $CD$ are parallel and equal (but we do not know whether sides $AC$ and $BD$ are parallel).




(a) Show that triangles $\triangle AMB$ and $\triangle CMD$ are congruent.

(b) Show that sides $AC$ and $BD$ are indeed parallel and therefore $ABCD$ is a parallelogram.

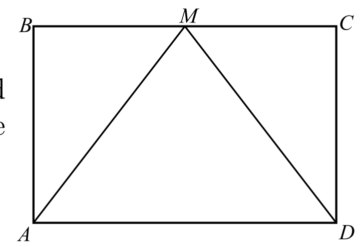
**29.** Let $ABCD$ be a parallelogram, and let $BE, CF$ be perpendiculars from $B, C$ to the line $AD$.




(a) Show that triangles $\triangle ABE$ and $\triangle DCF$ are congruent.

(b) Show that the area of parallelogram is equal to height$\times$base, i.e. $BE \times AD$.

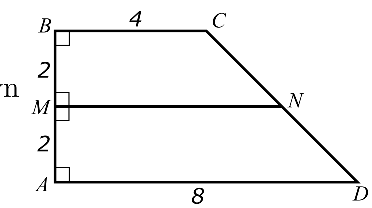
**30.** In the figure to the right, $ABCD$ is a rectangle, and $M$ is the midpoint of $BC$. Prove that then triangle $AMD$ is isosceles.




**31.** (a) Show that if in a quadrilateral $ABCD$ diagonals bisect each other (i.e., intersection point is hte midpoint of each of the diagonals), then $ABCD$ is a parallelogram. [Hint: find some congruent triangles in the figure.]

(b) Show that if in a quadrilateral $ABCD$ diagonals bisect each other and are perpendicular, then it is a rhombus.

**32.** Find all lenghts, angles, and area in the figure shown to the right.



# 3 Math 6 — Prerequisites for Math 7

## 3.1 Program

- Basics of logic. Knights and knaves. NOT, AND, OR, IF.
- Sets. Notation. Union, intersection, complement. Cardinality.
- Factorials and permutations.
- Ruler and compass constructions: midpoint, perpendicular, bisector.
- Coordinates. Equation of the line.
- Distance between two points on a coordinate plane. Equation of the circle.
- Arithmetic sequence. Geometric sequence. Formula for the general term. Formula for the sum.

## 3.2 Problems

**1.** On the island of knights and knaves, you meet two inhabitants: Sue and Zippy. Sue says that Zippy is a knave. Zippy says, "I and Sue are knights." So who is a knight and who is a knave?

**2.** On the island of Knights and Knaves, you meet three inhabitants:Bozo, Carl and Joe. Bozo says that Carl is a knave. Carl tells you, 'Of Joe and I, exactly one is a knight.' Joe claims, 'Bozo and I are different.'

**3.** On the island of Knights and Knaves, a traveler meets two inhabitants: Carl and Bill. Bill says: "Carl is a Knave". Carl says: "If Bill is a Knight, then I am a Knight, too."

**4.** Prove that

$$\text{NOT}(A \text{ AND } B) \text{ is the same as } (\text{NOT } A) \text{ OR} (\text{NOT } B)$$

**5.** Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?

(a) $(A \text{ OR } B) \text{ AND} (A \text{ OR } C)$

(b) $A \text{ OR} (B \text{ AND } C)$.

**6.** If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...

**7.** Let us take the usual deck of cards. As you know, there are 4 suits, hearts, diamonds, spades and clubs, 13 cards in each suit.

Denote:

$H$=set of all hearts cards

$Q$=set of all queens

$R$=set of all red cards

Describe by formulas (such as $H \cap Q$) the following sets:

all red queens

all black cards

all cards that are either hearts or a queen

all cards other than red queens

How many cards are there in each set?

**8.** Let

$A$=set of all people who know French

$B$=set of all people who know German

$C$=set of all people who know Russian

Describe in words the following sets:
(a) $A \cap B$ (b)$A \cup (B \cap C)$ (c) $(A \cap B) \cup (A \cap C)$ (d) $C \cap \overline{A}$.

**9.** In a class of 25 students, 10 students know French, 5 students know Russian, and 12 know neither. How many students know both Russian and French?

**10.** Let $A = [1, 3] = \{x \mid 1 \leq x \leq 3\}$, $B = \{x \mid x \geq 2\}$, $C = \{x \mid x \leq 1.5\}$. Draw on the number line the following sets: $\overline{A}$, $\overline{B}$, $\overline{C}$, $A \cap B$, $A \cap C$, $A \cap (B \cup C)$, $A \cap B \cap C$.

**11.** Show that for two sets $A, B$, we have $|A \cup B| = |A| + |B| - |A \cap B|$.

**12.** A group of 6 club members always dine at the same round table in the club; there are exactly 6 chairs at the table. They decided that each day, they want to seat in a different order. Can they keep this for a year? Two years?

**13.** In a computer game, a wizard is more powerful than an orc, so when a wizard fights an orc, he has 60% chance of winning. If a wizard fights one by one a group of 5 orcs, what are the chances that he will defeat them all?

**14.** In how many ways can one arrange 5 books on a shelf?

**15.** Show how to find a midpoint of an interval using ruler and compass.

**16.** Show how to construct a bisector using ruler and compass.

**17.** Draw all points on the plane for which one has $x = y + 1$.

**18.** Point $M$ has coordinates $(5, 7)$.

(a) Find coordinates of the point $M_1$ obtained from M by reflection around the $x$-axis

(b) Find coordinates of the point $M_2$ obtained from $M$ by reflection around the diagonal line.

**19.** Draw the graphs of the following functions:

(a) $2x + 3y = 1$

(b) $2x - 1 = y$

(c) $y = |x| - 2$

**20.** Find the distance between points $(2, 4)$ and $(3, 7)$.

**21.** Write the equation of the circle with center at $(1, 1)$ and radius 5.

**22.** What are the first 2 terms for the arithmetic sequence $a_1, a_2, 9, 2, 5, \ldots$?

**23.** In arithmetic sequence $a_{10} = 131$ and $d = 12$. What is $a_1$?

**24.** In arithmetic sequence $a_5 = 27$ and $a_{27} = 60$. Find the first term and the common difference.

**25.** Find the sum of the first 100 terms of the arithmetic sequence if $a_1 = 10$ and $a_{100} = 150$.

**26.** What are the first 2 terms for the geometric sequence $a_1, a_2, 24, 36, 54, \ldots$?

**27.** A geometric sequence has 99 terms, and the first term is 12 and the last term is 48. What is the 50th term?

**28.** Compute

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{10}}$$

**29.** Find the infinite sum

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots.$$