GAME NUMBERS 5

1. Practical Theorem 1: MEX

Given a game position G, there are a certain number of positions it can move to - these are its movable positions. Each movable position of G has a nimber. The smallest nimber that's not included in this collection is called the minimum excluded, or mex, of the moves of G. So, for example, if G can move to positions of nimbers 0, 1, 3, and 5, then its mex is 2.

Theorem 1 (MEX)**.** *A game position's nimber is equal to the mex of its movable positions.*

So, a strategy to find nimbers of the positions of a game is to use charts or tables or diagrams and take notes of where each position can move to, and find the mex of the nimbers of its movable positions. You will always need to start at 0, the empty position; then, build things up one by one, starting from positions at very small numbers of possible pieces or moves.

2. Practical Theorem 2: Nim Addition

The last, and final, theorem of this course, is the rule for how to add nimbers.

Let the binary representation of an integer be an expression of that integer with a sequence of digits that represent powers of 2. So, for example, 1000 in binary is equal to 2^3, because it's got a 1 in the third-power position. This parallels our standard numbers, which are called decimal or base 10, where 1000 is equal to 10^3.

Let's count in binary to see what the first few numbers look like. I will use a subscript to represent the base, so 10_2 is the binary number 10, which is equal to 2_{10}, where 2_{10} is the normal 2.

$$\begin{gathered} 1_2 = 1_{10} \\ 10_2 = 2_{10} \\ 11_2 = 3_{10} \\ 100_2 = 4_{10} \\ 101_2 = 5_{10} \\ 110_2 = 6_{10} \\ 111_2 = 7_{10} \\ 1000_2 = 8_{10} \end{gathered}$$

As we have defined before, the nim sum of two nimbers is the nimber of the game that results from the sum of those nimbers as nim piles. So, $5 + 6$ is the nimber of the game $Nim(5, 6)$.

Theorem 2 (Nim Addition)**.** *The nim sum of two nimbers is equal to the sum of their binary representations without carries.*

Proof. To understand what 'without carries' means, here's an example in base 10: $8 + 2 = 0$, because $8 + 2$ adds to a 0 in the 1's place with a carry in the tens place, but we delete the carry.

So, in binary, $1 + 1 = 0$. Also $10 + 1 = 11$, $11 + 1 = 10$, and $11 + 10 = 1$. So far these are all familiar rules to you, these are equations we have proved.

The idea behind this theorem is to split the nim piles, in your imagination, into subpiles whose sizes are powers of 2. So, $Nim(3, 5)$ is imagined as $Nim(2 + 1, 4 + 1)$. Then $3 + 6 = 2 + 1 + 4 + 1 = 4 + 2$. This is equivalent to writing the numbers in binary and doing no-carry addition: the 1s place cancels out, but the 4s and 2s places end up in the final result. So, $11_2 + 110_2 = 101_2$, as expected. □