GAME NUMBERS 4

1. Initial Theorems

Theorem 1 (P position)**.** *If a position can move to a P position, then it is an N position. If a position has no moves to a P position, then it is a P position. Additionally, the* 0 *game, or the empty game, is a P position, and every other position is either P or N.*

Theorem 2 (Symmetry)**.** *Given any game position G, the game position $G + G$ is a P-position.*

Definition 1 (equivalence)**.** Two game positions G and H are said to be equivalent if $G + H$ is a P-position. In this case, we write $G = H$.

Theorem 3 (P=0)**.** *Any P-position is equivalent to the 0 position.*

Theorem 4 (0 addition)**.** *Given any game position G, we have $G + 0 = G$.*

Theorem 5 (Arithmetic Symmetry)**.** $G + G = 0$ *for any game position G.*

Theorem 6 (Arithmetic)**.** *Given any game positions A, B, C, we have:*
$A = B$ *if and only if* $A + C = B + C$.
$A = B$ *and* $B = C$ *implies* $A = C$.
$A + B = B + A$.
$(A + B) + C = A + (B + C)$.

2. One-Pile Nim

Recall the game of Nim. In Nim, one has piles of stones, and one can take as many stones from a single pile as one wants. What if there is only one pile? This is a trivial version of the game - the winning strategy is clearly to just take all the stones. So this game is easily understood. Or so it seems.

One-pile Nim with zero stones in it is the 0 game, which we call $Nim(0)$. It is a P position. Also, by Theorem 3, all P positions are equal to $Nim(0)$. Then, we can deduce that all P positions are therefore equal to eac other.

All game positions are P positions or N positions. If all P positions are equal to each other, then are all N positions equal to each other?

Recall that a position is an N position if the first player can win. In one-pile nim, the first player always wins as long as there are more than zero stones. So $Nim(x)$ is an N position for all $x > 0$. Are all these N positions equal to each other?

One can deduce, for example, that $Nim(1) \neq Nim(2)$. Also, $Nim(1) \neq Nim(3)$, $Nim(2) \neq Nim(3)$, etc. Ultimately, it's possible to prove that $Nim(x) = Nim(y)$ only if $x = y$. Therefore, different Nim piles are unequal.

Therefore, there are *many different* N positions, even though there is essentially only one P position.

3. Nim Theorem

Here's the kicker: one-pile Nim covers *all possible* N positions. In fact, together with $Nim(0)$ which covers P positions, one-pile Nim covers *all possible game positions.*

Theorem 7 (Nim Theorem)**.** *Given any game positioon G, if G is an N position, then $G = Nim(x)$ for some whole number $x \geq 0$.*

This theorem is also called the Sprague-Grundy theorem.

The number x that a game position is equivalent to is called its nimber.

Definition 2 (Nimber)**.** Given a game position G, if $G = Nim(x)$, then x is the nimber of G.

From here forward, we can just use numbers to represent nimbers. Since all our arithmetic is with games, when I write $G = 4$, take this to mean that G has nimber 4, or that $G = Nim(4)$.

4. More Arithmetic

$$1+4=5$$
$$1+5=4$$
$$1+6=7$$
$$1+7=6$$
$$1+8=9$$
$$1+9=8$$
$$2+4=6$$
$$7=6+1=4+2+1$$
$$6=4+2$$
$$5=4+1$$
$$4=4$$
$$3=2+1$$
$$2=2$$
$$1=1$$
$$0=0$$
$$7+6=?$$