July 24, 2020. **Math 9+. Geometry Revisited and more.**

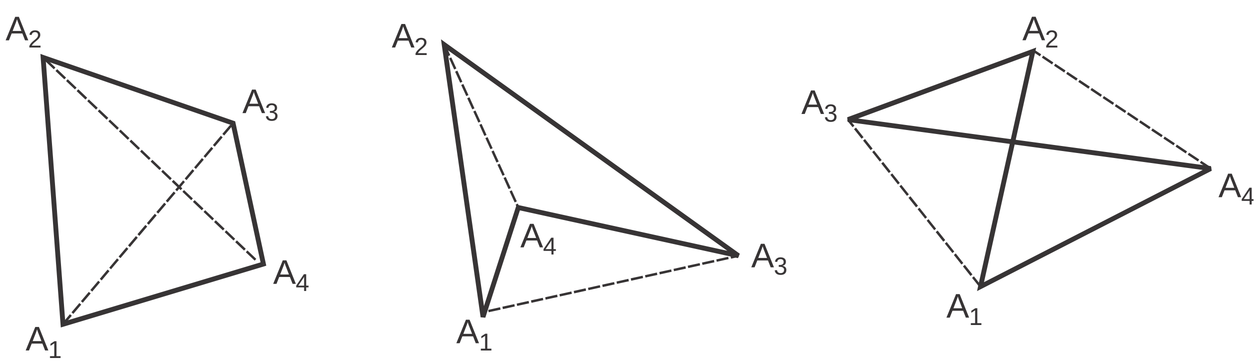
Collinearity and Concurrence. Quadrangles.

<u>Quadrangles. Varignon's theorem</u>.

Definition. A polygon is a geometrical figure consisting of a number of points (vertices) and an equal number of line segments (edges, or sides) connecting these points; if set of vertices is cyclically ordered, then no three successive points with the corresponding segments are collinear. In other words, a polygon is a closed broken line in a plane. A polygon with n vertices is called n-gon.

Definition. A simple polygon is a polygon whose edges do not intersect. If edges of a polygon do intersect, we obtain self-intersecting polygon, such as star polygon.

All triangles are simple polygons. This, however, is not the case for quadrangles (note, not every quadrangle is a quadrilateral!) and other n-gons. Two sides of a quadrangle that share a vertex are adjacent. Sides that do not have common vertex are opposite. Two vertices of a quadrangle that belong to a common side are adjacent; the other two are opposite. Diagonals are the lines joining the pairs of opposite vertices. In quadrangles $A_1A_2A_3A_4$ shown in the figure below, A_1A_3 and A_2A_4 are diagonals.

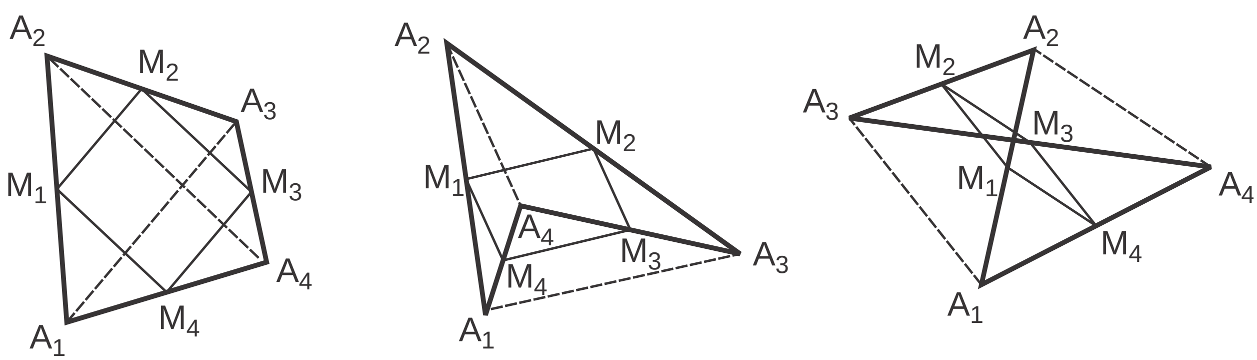


In a convex quadrangle, both diagonals are inside; in a concave (re-entrant) quadrangle one diagonal is inside and one outside; in a self-intersecting (crossed) quadrangle, both diagonals are outside (see figure).

The area of a convex quadrangle is the sum of the areas of the two triangles into which it is split by a diagonal:

$$S_{A_1A_2A_3A_4} = S_{A_1A_4A_3} + S_{A_3A_2A_1} = S_{A_4A_3A_2} + S_{A_2A_1A_4}$$

Note, that we chose to enumerate the vertices in such a way that for a convex quadrangle these vertices are named in a counterclockwise order for each triangle. If we now make the area of a triangle signed, that is, being positive, or negative, depending on whether the vertices of that triangle are named in a counterclockwise, or clockwise order, then the same formula would also hold for a concave quadrangle.



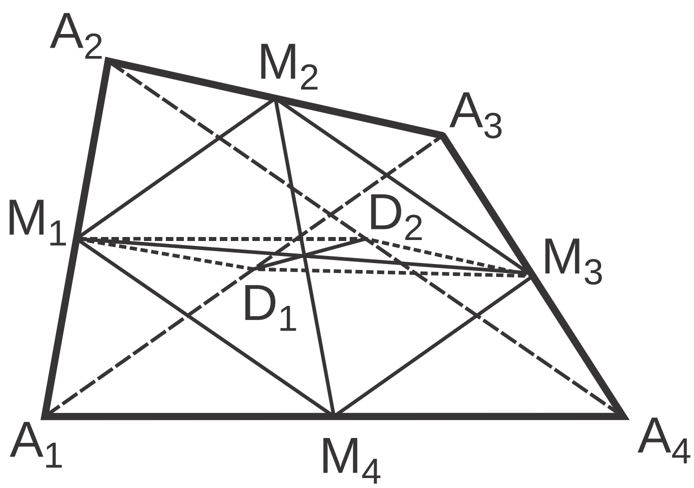
Exercise. Will the same formula expressing the area of a quadrangle through the (signed) areas of the two triangles into which it is split by a diagonal also hold for a self-intersecting quadrangle?

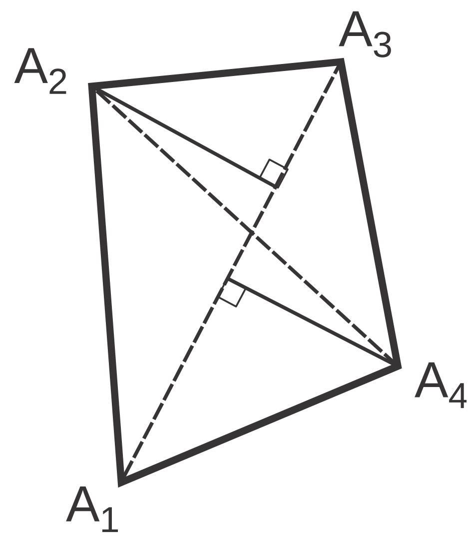
Theorem (Varignon). The figure formed by connecting the midpoints of the sides of a quadrangle is a parallelogram, and its area is half that of the quadrangle. It is often called the Varignon parallelogram of a quadrangle.

Exercise. Prove the above theorem.

Theorem (on concurrence). The segments joining the midpoints of pairs of opposite sides of a quadrangle and the segment joining the midpoints of the diagonals are concurrent and bisect one another.

Exercise. Prove the above theorem.

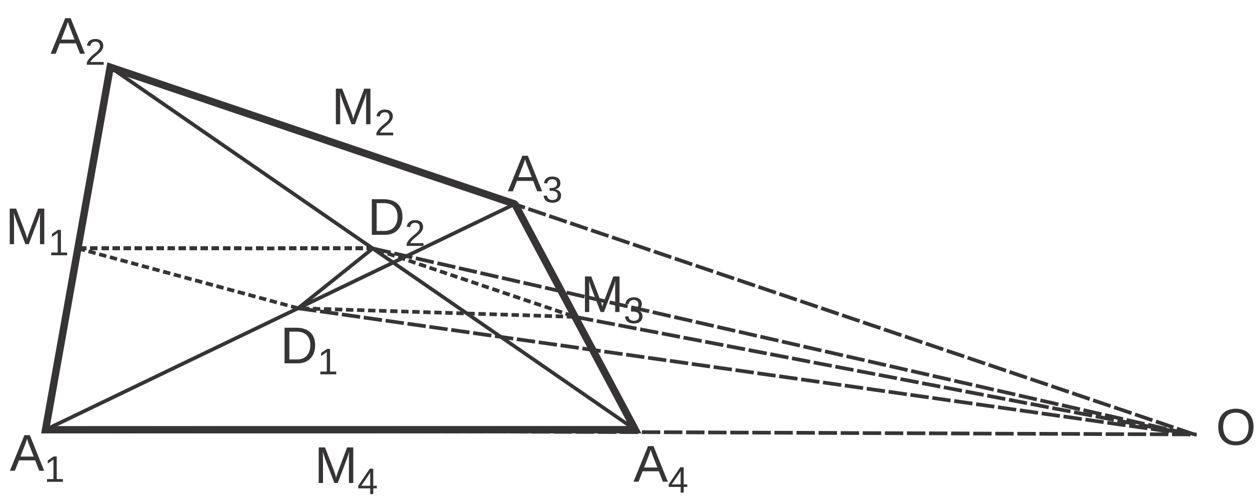




Theorem. If a diagonal divides a quadrangle into two triangles of equal area, it bisects the other diagonal. Conversely, if one diagonal of a quadrangle bisects the other, it bisects the area of the quadrangle.

Exercise. Prove the above theorem.

Theorem. If opposite sides, A_1A_4 and A_2A_3, of a quadrangle $A_1A_2A_3A_4$ (extended to) meet at point O and D_1 and D_1 are the midpoints of the diagonals A_1A_3 and A_2A_4, then the area of the triangle OD_1D_2 is a quarter of the area of the quadrangle, $S_{OD_1D_2} = \frac{1}{4} S_{A_1A_2A_3A_4}$



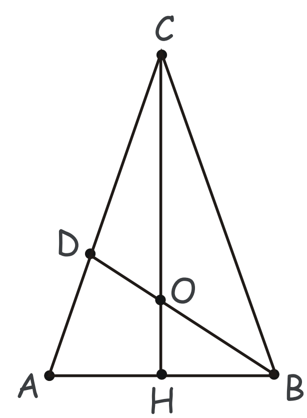
Exercise. Prove the above theorem.

Concurrence and mass points (method of the center of mass).

1. **Problem**. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).

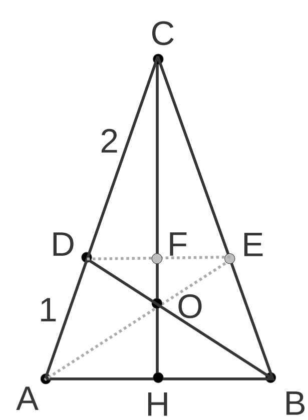
   Solution. Load vertices A, B and C with equal masses, m. Then, the center of mass (COM) of the three masses is at the intersection of the three medians, because it has to belong to each segment connecting the mass at the vertex of the triangle with the COM of the other two masses, i.e. the middle of the opposite side. COM this belongs to all three medians and is the centroid, O of the triangle. It divides each median in the 2:1 ratio because it is a COM of mass m at the vertex and a mass $2m$ at the middle of the opposite side.

2. **Problem**. In isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



   Solution.

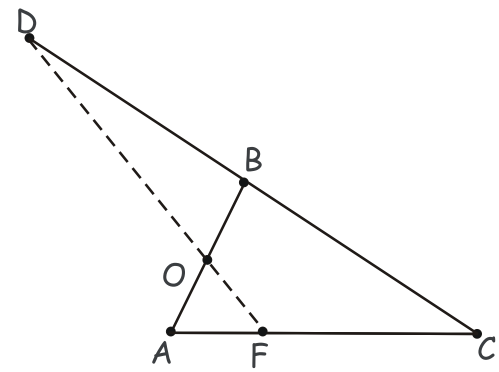
   a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio



is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.

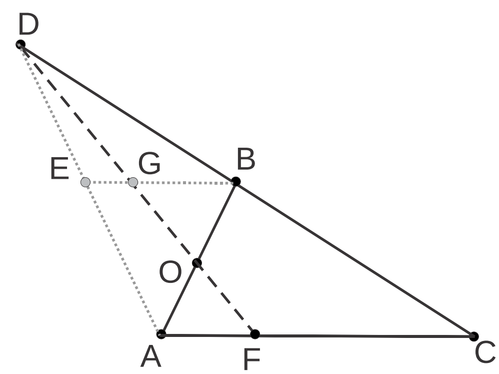
b. Using the Method of the Center of Mass. Load vertices A, B and C with masses $2m$, $2m$, and m, respectively. Then, H is the COM of masses at A and B, and D is the COM of masses at A and C, and O is the COM of all 3 masses in the vertices of the triangle ABC. Therefore, $|OC|:|OH| = (2m + 2m):m = 4:1$, $|OH|:|CH| = 1:5$.

3. **Problem**. Point D belongs to the continuation of side CB of the triangle ABC such that $|BD| = |BC|$. Point F belongs to side AC, and $|FC| = 3|AF|$. Segment DF intercepts side AB at point O. Find the ratio $|AO|:|OB|$.

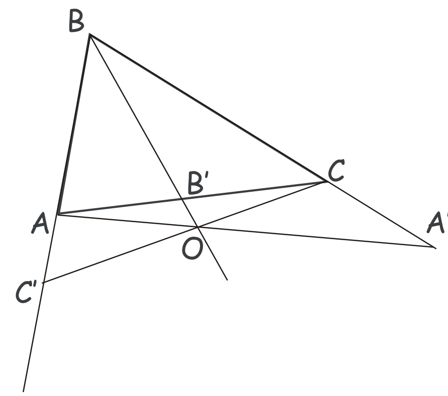


Solution.

a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment BE parallel to AC, $BE||AC$, where E belongs to the side AD of the triangle ACD. BE is the mid-line of the triangle ACD, and, by Thales, also of AFD and FDC. Therefore, $|EG| = \frac{1}{2}|AF|$, $|GB| = \frac{1}{2}|FC|$ and $|EB| = \frac{1}{2}|AC|$, so $\frac{|BG|}{|EG|} = \frac{|FC|}{|AF|} = 3$. On the other hand, again, by Thales, or, noting similar triangles $AOF \sim BOG$, $\frac{|AO|}{|OB|} = \frac{|AF|}{|GB|} = 2\frac{|AF|}{|AC|} = \frac{2}{3}$.



b. Using the Method of the Center of Mass. Load vertices A, C and D with masses $3m$, m and m, respectively. Then, F is the center of mass (COM) of A and C, B is the COM of D and C, and O is the COM of the triangle ACD, $|AO|:|OB| = (m + m):3m = 2:3$.



Theorem (Extended Ceva). Segments (Cevians) connecting vertices A, B and C, with points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are concurrent if and only if,

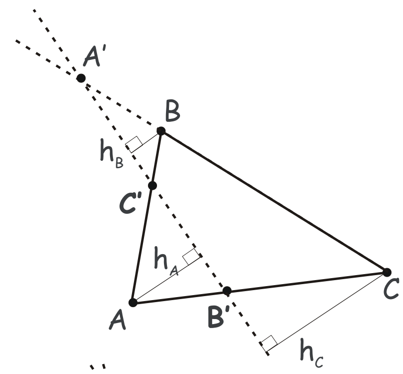
$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$$

Proof. We have already proven this theorem for the case when points A', B' and C' lie on the sides, but not on the lines extending the sides as it is shown in the figure. Let us now consider this latter case. Let us first load points A', B and C' with masses $m_{A'}$, m_B and $m_{C'}$ such that point A is the center of mass for m_B and $m_{C'}$, $m_B|AC'| = m_{C'}|AB|$, and point C is the COM for $m_{A'}$ and m_B, $m_{A'}|BC| = m_B|A'C|$. Then, the COM of all three masses at the vertices of the triangle $A'BC'$ is at the point O, which is the intersection of AA' and CC'. Let BO cross side AC at point B'. Adding mass to vertex B would move the COM of the three masses along line BO, because the COM of the initial 3 masses is at O. Let us add another mass m_B to vertex B, so that the total mass at this vertex is $2m_B$.The resulting system of masses then has the same COM as two masses, $m_B + m_{A'}$ and $m_B + m_{C'}$ at points A and C, respectively. This COM is common to AC and BO, and therefore is at point B', so $(m_B + m_{A'})|AB'| = (m_B + m_{C'})|B'C|$. Hence, we obtain,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = \frac{1}{1+\frac{m_{C'}}{m_B}}\left(1+\frac{m_{A'}}{m_B}\right)\frac{m_B + m_{C'}}{m_B + m_{A'}} = 1$$

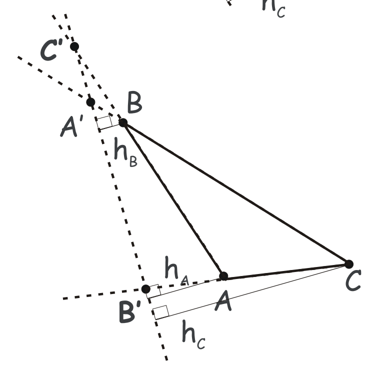
Theorem (Menelaus). Points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are collinear (belong to the same line) if and only if,

$$\frac{|A'B|}{|A'C|}\frac{|B'C|}{|B'A|}\frac{|C'A|}{|C'B|} = 1$$



Menelaus's theorem provides a criterion for collinearity, just as Ceva's theorem provides a criterion for concurrence.

Proof (similarity). The statement could be proven with, or without using the method of point masses.



First, assume the points are collinear and consider rectangular triangles obtained by drawing perpendiculars onto the line A'B'. Using their similarity, one has

$$\frac{|A'B|}{|A'C|} = \frac{h_B}{h_C}, \frac{|B'C|}{|B'A|} = \frac{h_C}{h_A}, \frac{|C'A|}{|C'B|} = \frac{h_A}{h_B}$$

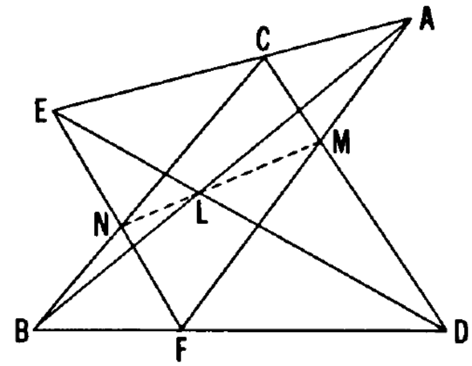
Wherefrom the statement of the theorem is obtained by multiplication (Coxeter & Greitzer).

Proof (point masses). Alternatively, let us load points A, A' and C in the upper Figure with the point masses m_1, m_2 and m_3, respectively. We select m_1, m_2 and m_3 such that B' is the COM of $m_1(A)$ and $m_3(C)$, and B is the COM of $m_2(A')$ and $m_3(C)$. The COM of all 3 masses belongs to both segments AB and $A'B'$, which means that it is at point C'. Then,

$$\frac{|A'B|}{|A'C|} = \frac{m_3}{m_2 + m_3}, \frac{|B'C|}{|B'A|} = \frac{m_1}{m_3}, \frac{|C'A|}{|C'B|} = \frac{m_2 + m_3}{m_1}$$

Wherefrom the Menelaus theorem is obtained by multiplication. The case shown in the lower figure is considered in a similar way.

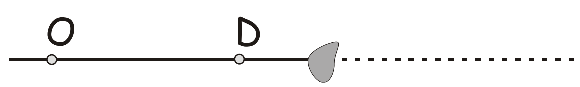
Theorem (Pappus). If A, C, E are three points on one line, B, D and F on another, and if three lines, AB, CD, EF, meet DE, FA, BC, respectively, then the three points of intersection, L, M, N, are collinear.



This is one of the most important theorems in planimetry, and plays important role in the foundations of projective geometry. There are a number of ways to prove it. For example, one can consider five triads of points, LDE, AMF, BCN, ACE and BDF, and apply Menelaus theorem to each triad. Then, appropriately dividing all 5 thus obtained equations, we can obtain the equation proving that LMN are collinear, too, also by the Menelaus theorem. However, one can prove the Pappus theorem directly, using the method of point masses.

Instead of simply proving the theorem, consider the following problem.

Problem. Using only pencil and straightedge, continue the line to the right of the drop of ink on the paper without touching the drop.



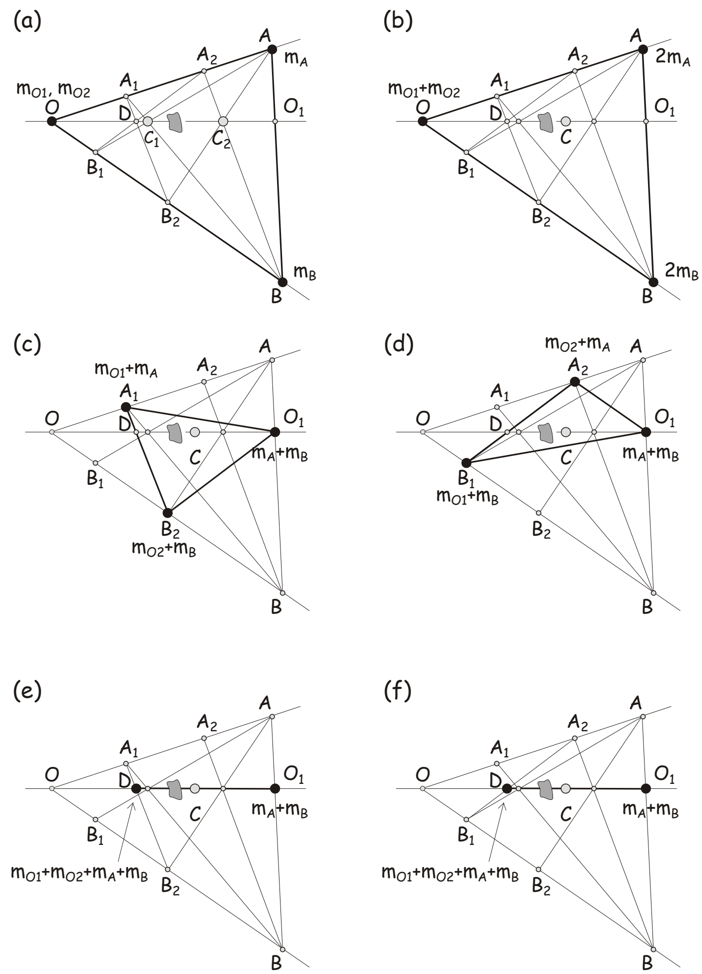
Solution by the Method of the Center of Mass.

Construct a triangle OAB, which encloses the drop, and with the vertex O on the given line (OD). Let O_1 be the crossing point of (OD) and the side AB. Let us now load vertices A and B of the triangle with point masses m_A and m_B, such that their center of mass (COM) is at the point O_1. Then, each point of the (Cevian) segment OO_1 is the center of mass of the triangle OAB for some point mass m_O loaded on the vertex O. The (Cevian) segments from vertices A and B, which pass through the center of mass of the triangle C, connect each of these vertices with the center of mass of the other two vertices on the opposite side of the triangle, OB and OA, respectively.

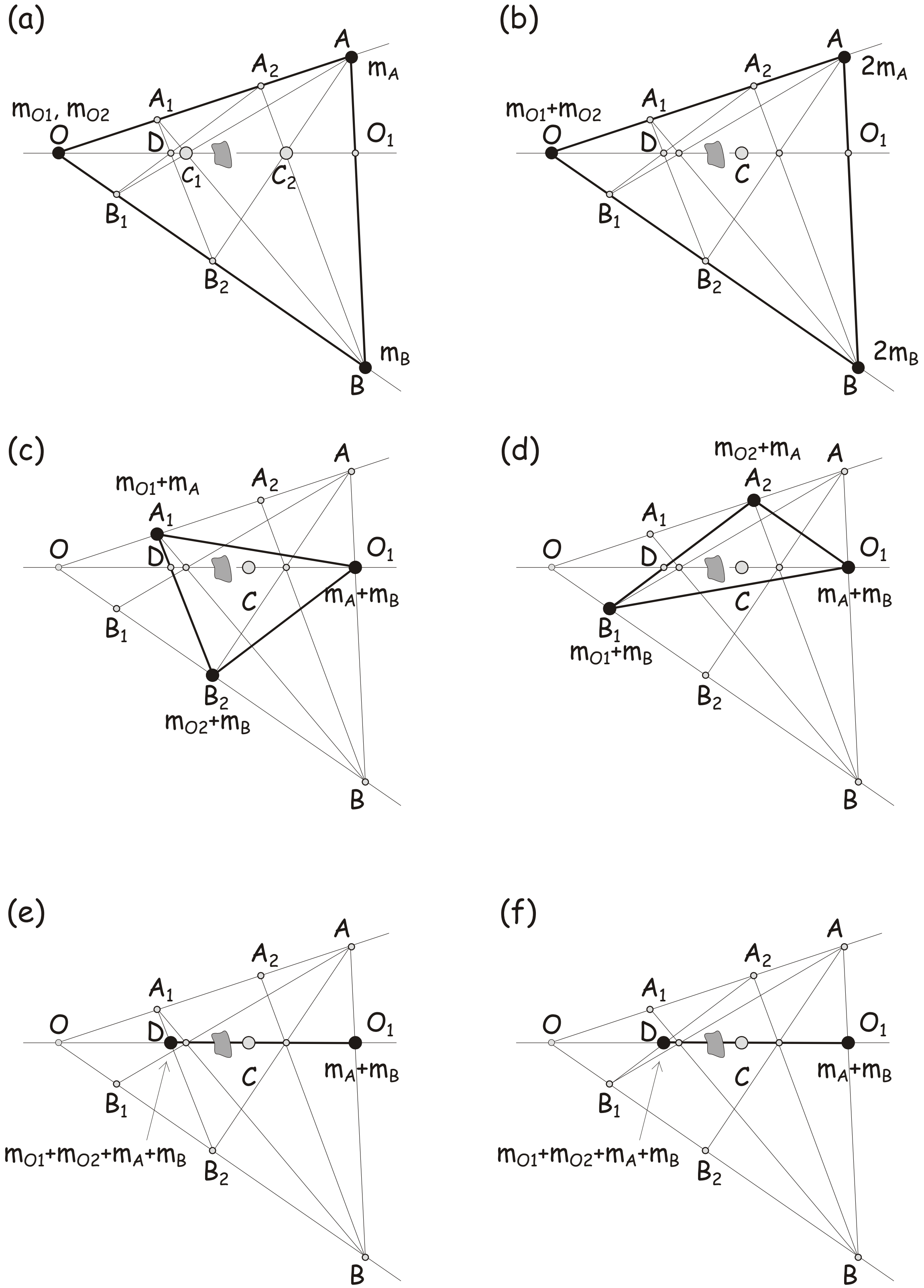
For the mass m_{O1} loaded on the vertex O, the center of mass of the triangle is C_1, and the centers of mass of the sides OA and OB are A_1 and B_1, respectively.

Similarly, C_2, A_2 and B_2 are those for the mass m_{O2} on the vertex O. The center of mass of the side AB is always at the point O_1, independent of mass m_O.

If we can show that segments A_1B_2 and A_2B_1 cross the given line (OD) at the same point, D, then our problem is solved, as we can draw Cevians BA_2 and AB_2, whose crossing points are on the segment OO_1 on the other side of the drop, by sequentially drawing Cevians BA_1 and AB_1 and segments A_1B_2, B_1A_2, Figure 1(a).



Let us load vertices O, A and B with masses $m_{O1}+ m_{O2}$, $2m_A$ and $2m_B$, respectively, Figure 1(b). The center of mass of OAB is now at some point C, in-between C_1 and C_2 (actually, it is not important where it is on the line OO_1). Let us now move point masses m_{O1} and m_A to their center of mass A_1 on the side OA, m_{O2} and m_B to their center of mass B_2 on the side OB, and m_A and m_B to their center of mass O_1 on the side AB. Now masses are at the vertices of the triangle $A_1B_2O_1$ with the same center of mass, C, Figure 1(c). Consequently, the crossing point D of segments A_1B_2 and OO_1 is the center of mass for masses $m_{O1}+m_A$ and $m_{O2}+m_B$ placed at points A_1 and B_2, respectively. Point C then is the center of mass for $m_{O1}+m_{O2}+m_A+m_B$ at point D and m_A+m_B at point O_1, Figure 1(e). Repeating similar arguments for the triangle $A_2B_1O_1$, Figure 1(d,f), we see that point D is also the crossing point of segments A_1B_2 and OO_1. Therefore, D is the crossing point of all three segments, A_1B_2, A_2B_1 and OO_1, which completes the proof.


(a)
A
m_A
A_2
A_1
m_{O1}, m_{O2}
O
D
O_1
C_1
C_2
B_1
B_2
m_B
B
(b)
A
$2m_A$
A_2
A_1
$m_{O1}+m_{O2}$
O
D
O_1
C
B_1
B_2
$2m_B$
B
(c)
A
A_2
$m_{O1}+m_A$
A_1
O
D
O_1
C
m_A+m_B
B_1
B_2
$m_{O2}+m_B$
B
(d)
$m_{O2}+m_A$
A
A_2
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D
O_1
C
m_A+m_B
B_1
$m_{O1}+m_B$
B_2
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(e)
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m_A+m_B
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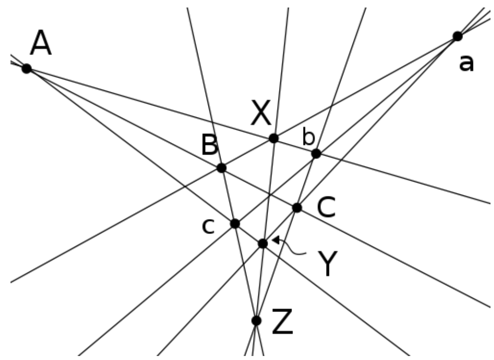
$m_{O1}+m_{O2}+m_A+m_B$
B_2
B
(f)
A
A_2
A_1
O
D
O_1
C
m_A+m_B
B_1
$m_{O1}+m_{O2}+m_A+m_B$
B_2
B

The Pappus hexagon formulation. If the six vertices of a hexagon lie alternately on two lines, then the three points of intersection of pairs of opposite sides are collinear.

The dual of Pappus theorem states that given one set of concurrent lines A, B, C, and another set of concurrent lines a, b, c, then the lines x, y, z defined by pairs of points resulting from pairs of intersections A∩b and a∩B, A∩c and a∩C, B∩c and b∩C are concurrent. (Concurrent means that the lines pass through one point.)



The Pappus configuration is the configuration of 9 lines and 9 points that occurs in Pappus's theorem, with each line meeting 3 of the points and each point meeting 3 lines. This configuration is self dual. (http://en.wikipedia.org/wiki/Pappus's_hexagon_theorem; http://www.cut-the-knot.org/pythagoras/Pappus.shtml).