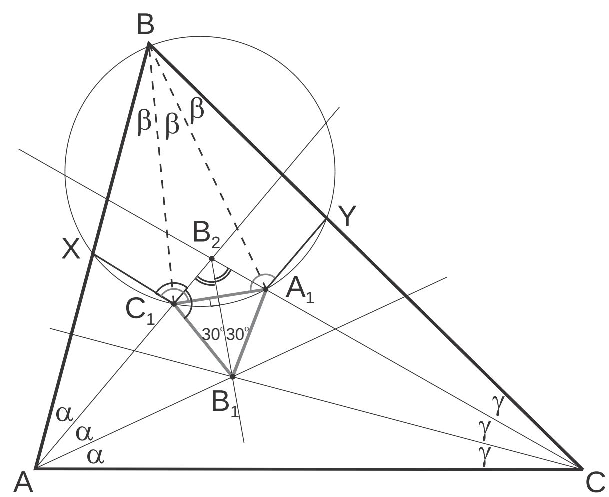
July 17, 2020. **Math 9+. Geometry Revisited and more.**

Collinearity and Concurrence. Morley theorem.

<u>Morley's theorem</u>.

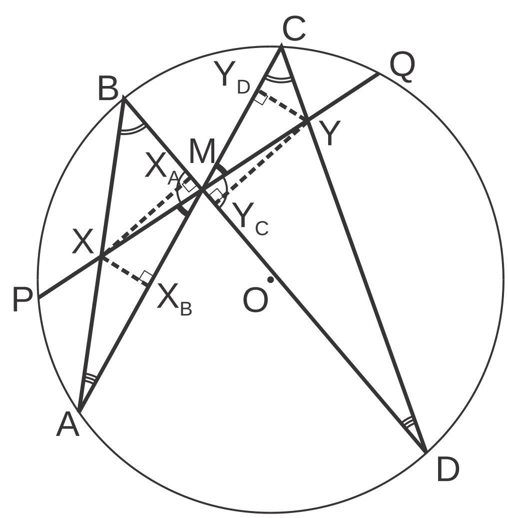
Definition. Three lines trisecting angle $\angle ABC$ into three equal angles are called trisectors of $\angle ABC$.

Theorem. The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle.



Proof. Let the adjacent trisectors of angles BAC and BCA meet at a point B_1 and the other two, non-adjacent trisectors meet at point B_2 (see figure). Then, B_1 is the incenter of the triangle AB_2C and $B_2B_1$is the bisector of $\angle AB_2C$. Let us now construct an equilateral triangle $A_1B_1C_1$ where A_1 and C_1 belong to the non-adjacent trisectors, CB_2 and AB_2, respectively. In order to prove the theorem, we must prove that BC_1 and BA_1 are the trisectors of the angle ABC.

Exercise. Complete the above proof.



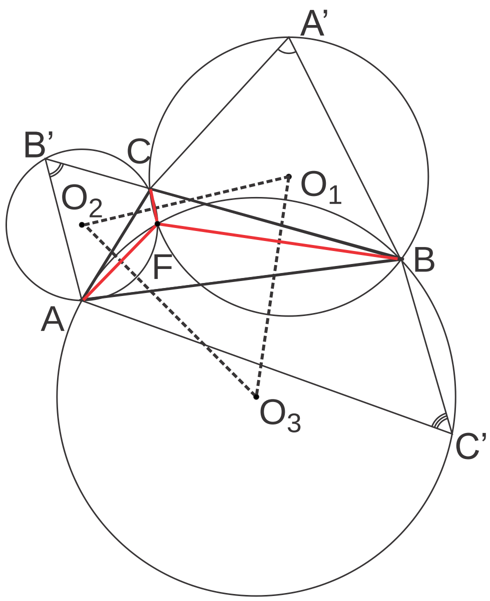
Exercise. Prove the following theorem (butterfly).

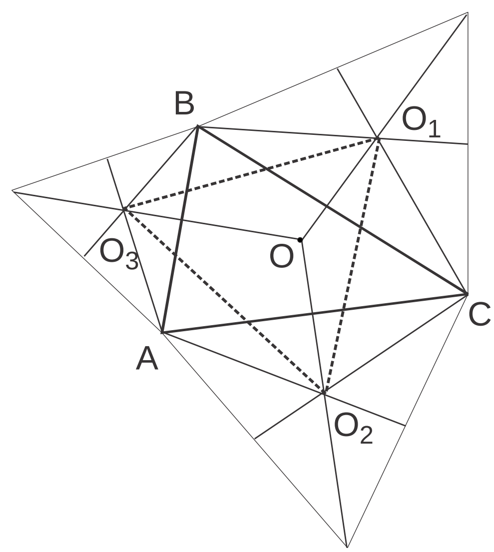
Theorem (butterfly). Through the midpoint M of a chord PQ of a circle, any other two chords, AC and BD are drawn. If chords AB and CD meet PQ at points X and Y, then M is the midpoint of XY.

Proof. Consider the figure.

Napoleon triangles.

Theorem (butterfly). If triangles are erected externally on the sides of an arbitrary triangle so that sum of the “remote” angles of these three triangles is 180°, then the circumcircles of these three triangles have a common point.





Proof. Consider the figure.

Theorem. Three equilateral triangles are erected externally on the sides of an arbitrary triangle ABC. Then, the triangle $O_1O_2O_3$ obtained by connecting the centers of these equilateral triangles is also an equilateral triangle (Napoleon’s triangle, see Figure).

Solution. Denote |AB| = c, |BC| = a, |AC| = b. Let us find the side $|O_2O_3|$.
Express $\overrightarrow{O_2O_3} = \overrightarrow{AO_3} - \overrightarrow{AO_2}$, or, $\overrightarrow{O_2O_3} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{C'O_3} - \frac{1}{2}\overrightarrow{AC} - \overrightarrow{B'O_2}$.

Note, that $\left|\overrightarrow{B'O_2}\right| = b\frac{\sqrt{3}}{6}$, and $\left|\overrightarrow{C'O_3}\right| = c\frac{\sqrt{3}}{6}$. Also, $\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right) = bc\cos\alpha$,
$\left(\overrightarrow{AB}\cdot\overrightarrow{B'O_2}\right) = \left(\overrightarrow{AC}\cdot\overrightarrow{C'O_3}\right) = bc\frac{\sqrt{3}}{6}\cos(90^\circ+\alpha) = -\frac{1}{2\sqrt{3}}bc\sin\alpha$, and

$\left(\overrightarrow{C'O_3} \cdot \overrightarrow{B'O_2}\right) = \frac{1}{12} bc \cos(180^\circ - \alpha) = -\frac{1}{12} bc \cos\alpha$, where $\alpha = \widehat{BAC}$. Then,

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{C'O_3}\right|^2 + \frac{1}{4}\left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{B'O_2}\right|^2 - \frac{1}{2}\left(\overrightarrow{AB} \cdot \overrightarrow{AC}\right)$$

$$-\left(\overrightarrow{AB} \cdot \overrightarrow{B'O_2}\right) - \left(\overrightarrow{AC} \cdot \overrightarrow{C'O_3}\right) - 2\left(\overrightarrow{C'O_3} \cdot \overrightarrow{B'O_2}\right), \text{ or,}$$

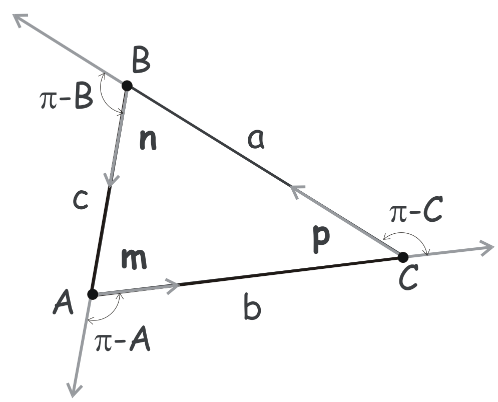
$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{4}\left(c^2 + \frac{1}{3}c^2 + b^2 + \frac{1}{3}b^2 - 2bc\cos\alpha + \frac{4}{\sqrt{3}} bc \sin\alpha + \frac{2}{3} bc\cos\alpha\right),$$

$$\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{3}c^2 + \frac{1}{3}b^2 - \frac{1}{3}bc\cos\alpha + \frac{1}{\sqrt{3}} bc \sin\alpha.$$

Now, using the Law of cosines, $2bc\cos\alpha = b^2 + c^2 - a^2$, and the Law of sines, $\sin\alpha = \frac{a}{2R}$, where R is the radius of the circumcircle, we obtain $\left|\overrightarrow{O_2O_3}\right|^2 = \frac{1}{6}a^2 + \frac{1}{6}b^2 + \frac{1}{6}c^2 + \frac{abc}{2\sqrt{3}R}$. Obviously, the same expression holds for the sides $|O_1O_3|$ and $|O_1O_2|$. Hence, triangle $O_1O_2O_3$ is equilateral.

Problem. Let A, B and C be angles of a triangle ABC.

a. Prove that $\cos A + \cos B + \cos C \leq \frac{3}{2}$.
b. *Prove that for any three numbers, m,n,p, $2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2$



Solution. Let vectors $\vec{m}, \vec{n}, \vec{p}$ be parallel to $\overrightarrow{AC}$, $\overrightarrow{BA}$ and $\overrightarrow{CB}$, respectively, as in the Figure. Then,

$$(\vec{m} + \vec{n} + \vec{p})^2 = m^2 + n^2 + p^2 - 2mn\cos A - 2np\cos B - 2mp\cos C$$

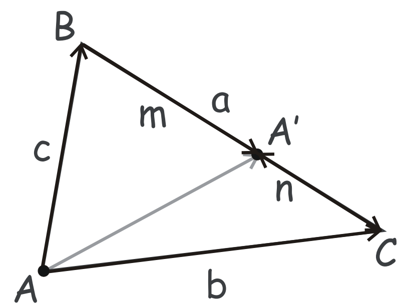
wherefrom immediately follows that,

$$2mn\cos A + 2np\cos B + 2pm\cos C \leq m^2 + n^2 + p^2.$$

The statement in part (a) follows from the above for $m = n = p = 1$.

Problem. Point A' divides the side BC of the triangle ABC into two segments, BA' and $A'C$, whose lengths have the ratio $|BA'|:|A'C| = m:n$. Express vector $\overrightarrow{AA'}$ via vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the length of the Cevian AA' if the sides of the triangle are $|AB| = c$, $|BC| = a$, and $|AC| = b$.

Solution. It is clear from the Figure, that $\overrightarrow{BA'} = \frac{m}{n}\overrightarrow{A'C} = \frac{m}{m+n}\overrightarrow{BC}$, and $\overrightarrow{CA'} = \frac{n}{m+n}\overrightarrow{CB} = \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right)$. Therefore,



$\overrightarrow{AA'} = \overrightarrow{AC} + \overrightarrow{CA'} = \overrightarrow{AC} + \frac{n}{m+n}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}$.

Or, we can obtain the same result as

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + \frac{m}{m+n}\left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}.$$

For the length of the segment AA' we have,

$|AA'|^2 = \overrightarrow{AA'}^2 = \left(\frac{n}{m+n}\overrightarrow{AB} + \frac{m}{m+n}\overrightarrow{AC}\right)^2 = \frac{n^2c^2+m^2b^2+(nm)2bc\cos\widehat{BAC}}{(m+n)^2}$. Using the Law of cosines, we write $2bc\cos\widehat{BAC} = b^2 + c^2 - a^2$, and obtain the final result,

$$|AA'|^2 = \frac{(n^2+nm)c^2+(m^2+nm)b^2-(mn)a^2}{(m+n)^2} = \frac{mb^2+nc^2}{m+n} - \frac{mna^2}{(m+n)^2}.$$

Or, equivalently, $(m+n)|BB'|^2 = mb^2 + nc^2 - \frac{mna^2}{m+n}$.

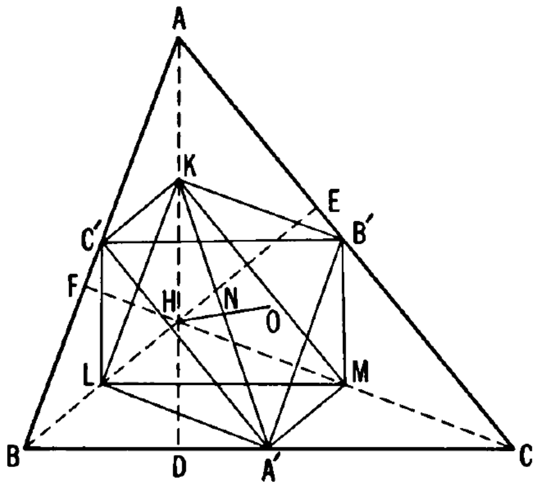
Substituting $m + n = a$, we obtain the Stewart's theorem (Coxeter, Greitzer, exercise 4 on p. 6).

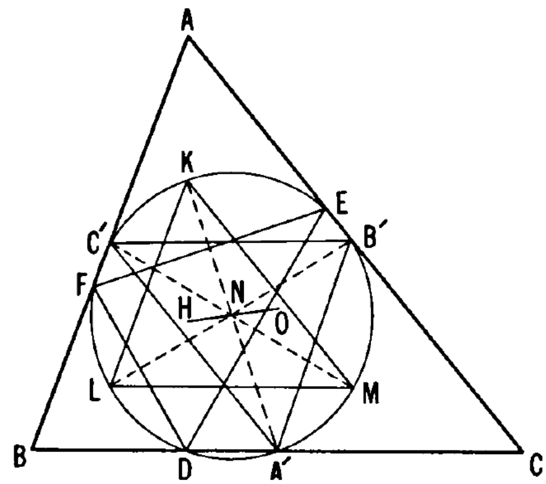
<u>If AA' is a median</u>, then $|BA'|:|A'C| = 1:1$, i.e. $m = n = 1$, and we have, $\overrightarrow{AA'} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$, $|AA'|^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$ (AA' is a median).

If AA' is a bisector, $|BA'|:|A'C| = c:b$, i.e. $m = c, n = b$, and we obtain $\overrightarrow{AA'} = \frac{b}{b+c}\overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{AC}$, as well as $|AA'|^2 = \frac{b^2c+c^2b}{b+c} - \frac{bca^2}{(b+c)^2} = bc\left(1 - \frac{a^2}{(b+c)^2}\right)$ (AA' is a bisector).

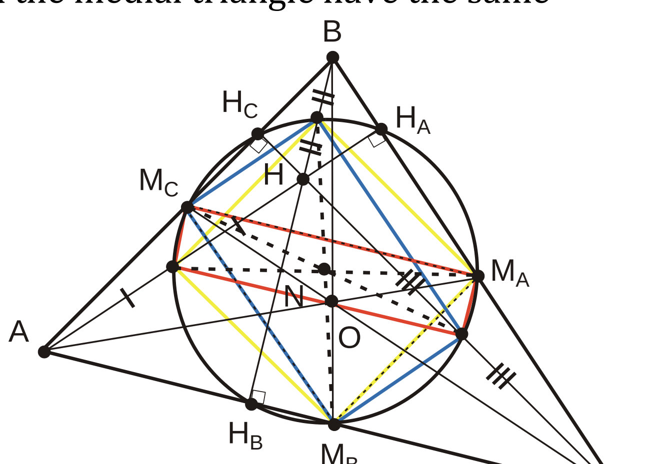
<u>The nine-points circle problem</u>.

Theorem. The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices lo the orthocenter, all lie on the same circle, of radius $\frac{1}{2}R$.



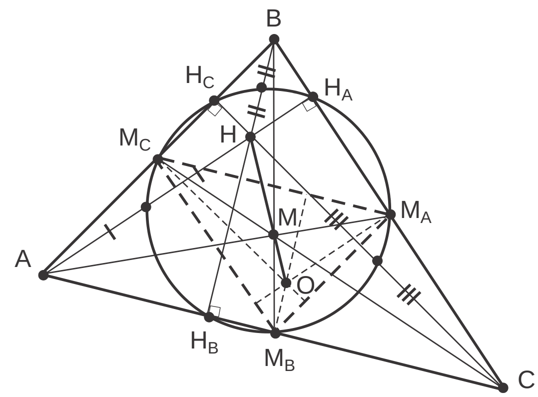


This theorem is usually credited to a German geometer Karl Wilhelm von Feuerbach, who actually rediscovered the theorem. The first complete proof appears to be that of Jean-Victor Poncelet, published in 1821, and Charles Brianson also claimed proving the same theorem prior to Feuerbach. The theorem also sometimes mistakenly attributed to Euler, who proved, as early as 1765, that the orthic triangle and the medial triangle have the same circumcircle, which is why this circle is sometimes called "the Euler circle". Feuerbach rediscovered Euler's partial result even later, and added a further property which is so remarkable that it has induced many authors to call the nine-point circle "the Feuerbach circle".



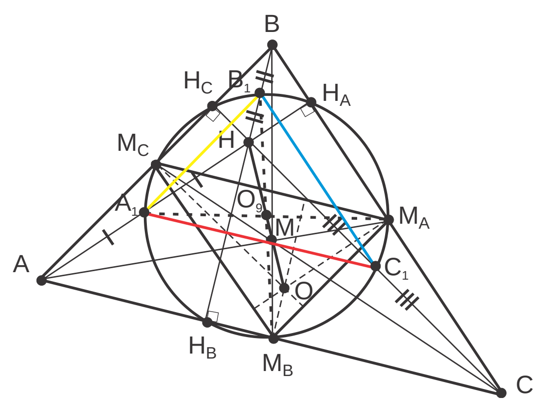
Proof. Consider rectangles formed by the mid-lines of triangle ABC and of triangles ABH, BCH and ACH.

Theorem. The orthocenter, H, centroid, M, and the circumcenter, O, of any triangle are collinear: all these three points lie on the same line, OH, which is called the Euler line of the triangle. The orthocenter divides the distance from the centroid to the circumcenter in 2: 1 ratio.



Proof. Note that the altitudes of the medial triangle$M_A M_B M_C$ are the perpendicular bisectors of the triangle ABC, so the orthocenter of $\Delta\ M_A M_B M_C$ is the circumcenter, O, of $\Delta\ ABC$. Now, using the property that centroid divides medians of a triangle in a 2:1 ratio, we note that triangles BMH and $M_B MO$ are similar, and homothetic with respect to point M, with the homothety coefficient 2.

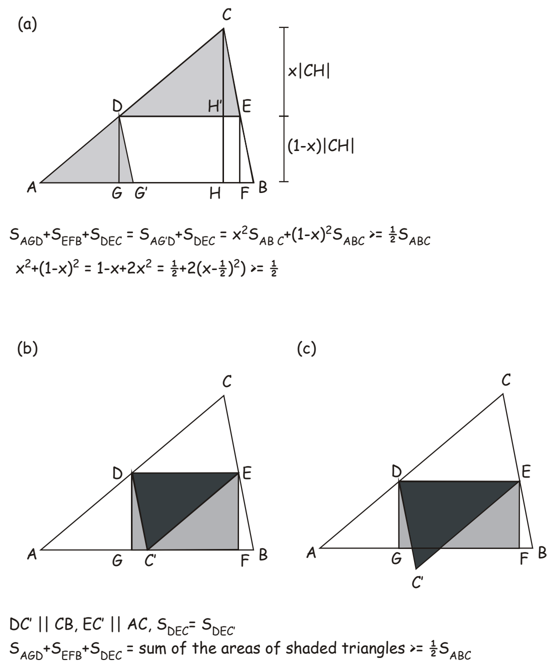
Theorem. The center of the nine-point-circle lies on the (Euler's) line passing through orthocenter, centroid, and circumcenter, midway between the orthocenter and the circumcenter.



Proof. Consider the figure. Note the colored triangle $A_1 B_1 C_1$, which is formed by medians of triangles ABH, BHC and CHA, and is therefore congruent to the medial triangle $M_A M_B M_C$, but rotated 180 degrees. The 9 points circle is the circumcircle for both triangles, which means that rotation by 180 degrees about the center O_9 of the 9 point circle moves $\Delta\ M_A M_B M_C$ onto $\Delta\ A_1 B_1 C_1$, and the orthocenter, O, of the $\Delta\ M_A M_B M_C$ onto the orthocenter, H, of the $\Delta\ A_1 B_1 C_1$.

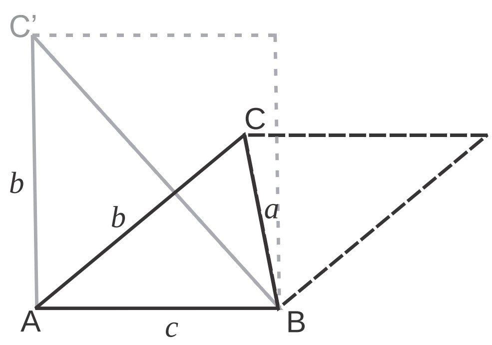
More problems.

Problem. Rectangle DEFG is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle DEFG?



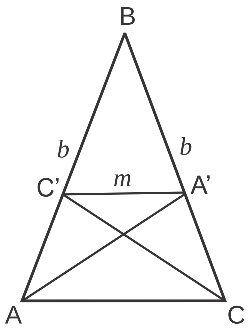
Solution. Notice similar triangles, $CDE \sim ABC$, wherefrom the vertical side of the rectangle is, $|DG| = |EF| = |CH| - |CH'| = \left(1 - \frac{|DE|}{|AB|}\right)|CH|$, so that the area of the rectangle is, $S_{DEFG} = |DE||DG| = |DE|\left(1 - \frac{|DE|}{|AB|}\right)|CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)|AB||CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)2S_{ABC}$. Using the geometric-arithmetic mean inequality, $\frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right) \leq \left(\frac{\frac{|DE|}{|AB|}+1-\frac{|DE|}{|AB|}}{2}\right)^2 = \frac{1}{4}$, where the largest value of the left side is achieved when $\frac{|DE|}{|AB|} = 1 - \frac{|DE|}{|AB|}$, and therefore $S_{DEFG} = \frac{1}{2}S_{ABC}$. There are a number of other possible solutions, some of which are shown in the figures.

Problem. Prove that for any triangle ABC with sides a, b and c, the area, $S \leq \frac{1}{4}(b^2 + c^2)$.



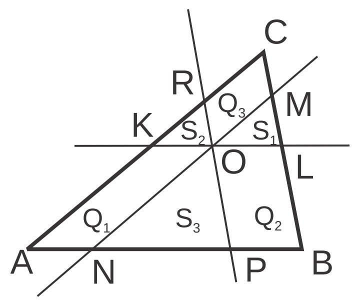
Solution. Notice that of all triangles with given two sides, b and c, the largest area has triangle ABC', where the sides with the given lengths, $|AB| = c$ and $|AC| = b$ form a right angle, $\widehat{BAC} = 90°$ (b is the largest possible altitude to side c). Therefore, $\forall \Delta ABC, S_{ABC} \leq S_{ABC'} = \frac{1}{2}bc \leq \frac{1}{2}\frac{b^2+c^2}{2}$, where the last inequality follows from the arithmetic-geometric mean inequality, $bc \leq \frac{b^2+c^2}{2}$ (or, alternatively, follows from $b^2 + c^2 - 2bc = (b - c)^2 \geq 0$.

Problem. In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).



Solution. From Thales proportionality theorem we have, $\frac{|AC|}{m} = \frac{|BC|}{|BA'|} = \frac{|BA'|+|A'C|}{|BA'|} = 1 + \frac{|A'C|}{|BA'|} = 1 + \frac{|AC|}{b}$, where we have used the property of the bisector, $\frac{|A'C|}{|BA'|} = \frac{|AC|}{|AB|} = \frac{|AC|}{b}$. We thus obtain, $|AC| = \frac{1}{\frac{1}{m}-\frac{1}{b}} = \frac{bm}{b-m}$.

Problem. Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$ (see Figure).



Solution. Denote $\frac{S_1}{S} = k_1, \frac{S_2}{S} = k_2, \frac{S_3}{S} = k_3$. Then, $\frac{S_1+S_2+Q_3}{S} = k_1 + k_2 + \frac{Q_3}{S} = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$, so, $Q_3 = 2S\sqrt{k_1k_2} = \sqrt{S_1S_2}$, $Q_2 = \sqrt{S_3S_1}$, $Q_1 = \sqrt{S_2S_3}$.