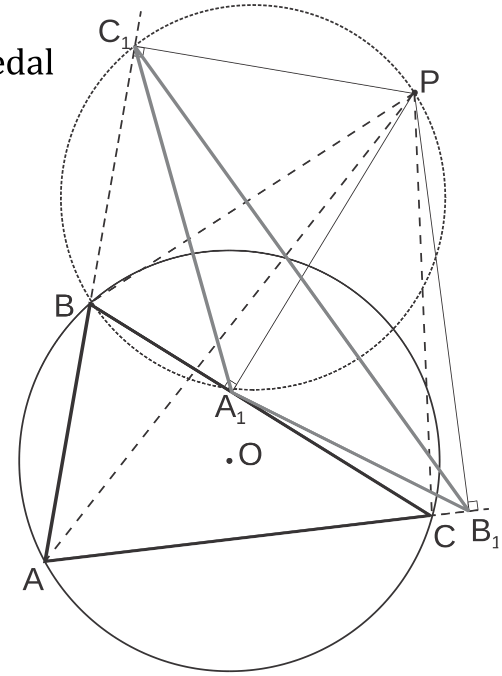
July 10, 2020. **Math 9+. Geometry Revisited and more.**

Collinearity and Concurrence

Degenerate pedal triangle. Simson lines.

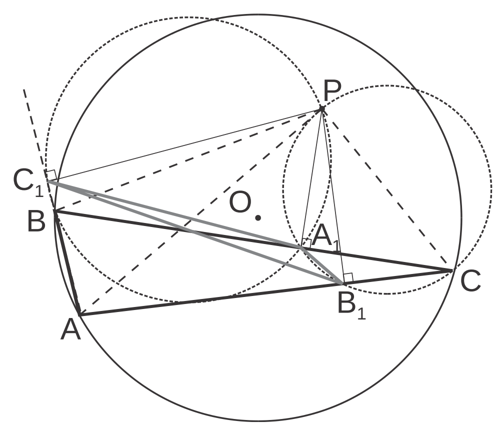


Definition. The feet of the perpendiculars from any point P of the plane, *inside or outside* a triangle (pedal point) to the three sides of that triangle are the vertices of the *pedal triangle*. **Note**: this includes a special case of P on a circumcircle considered below, which corresponds to a degenerate triangle.

Theorem. If the distances from a pedal point P of a triangle ABC to its vertices are $|AP| = x$, $|BP| = y$, and $|CP| = z$ and the sides of the triangle are $|AB| = c$, $|BC| = a$, and $|AC| = b$, then the sides of the pedal triangle are,

$|B_1C_1| = \frac{ax}{2R}$, $|C_1A_1| = \frac{by}{2R}$, and $|A_1B_1| = \frac{cz}{2R}$,

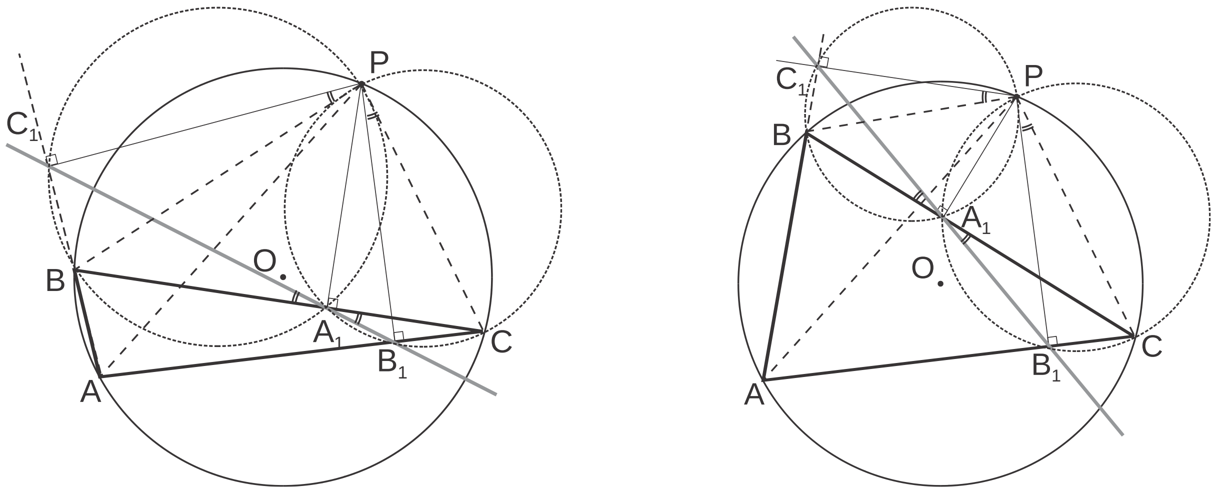
where R is the circumradius of the triangle ABC.



Exercise. Prove the above theorem for configurations of pedal triangle shown in the figures, with pedal point outside of triangle ABC. Is there any difference between the cases of obtuse and acute triangle ABC?

Proof. Consider the circumcircles of the triangles AC_1B_1, $C_1BA_1$and B_1A_1C. Using the extended sine theorem for ΔAC_1B_1, $\frac{|B_1C_1|}{\sin A} = |AP|$ and $\frac{a}{\sin A} = 2R$, so $|B_1C_1| = a\frac{|AP|}{2R}$. For ΔC_1BA_1, $\frac{|A_1C_1|}{\sin(180-B)} = |BP|$; using $\frac{b}{\sin B} = 2R$, $|A_1C_1| = b\frac{|BP|}{2R}$.

Theorem. The feet of the perpendiculars from a point to the sides of a triangle are collinear if and only if the point lies on the circumcircle. The line they lie on is called Simson line.



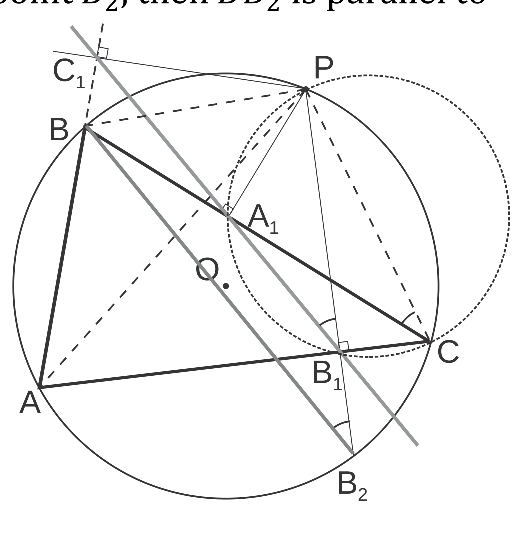
Proof. Consider the figure. Note that $\angle C_1PB_1 = \angle BPC = 180° - \angle BAC$, wherefrom follows that $\angle BPC_1 = \angle B_1PC$. Furthermore, $\angle BA_1C_1 = \angle BPC_1$ and $\angle B_1A_1C = \angle B_1PC$ by the inscribed angle theorem. The collinearity of C_1, A_1, and B_1then follows from the congruence of angles $\angle BA_1C_1$ and $\angle B_1A_1C$.

Exercise. Does the proof of the above theorem require any modification when $\Delta\, ABC$ is obtuse?

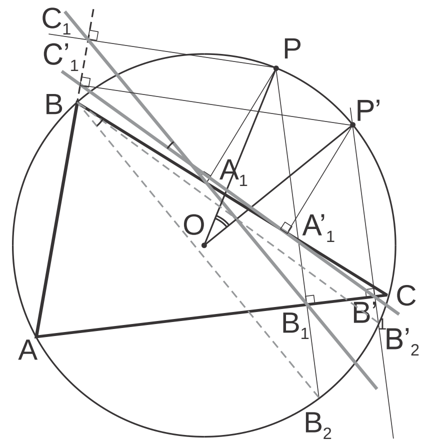
Exercise. Is there a point on the circle that has side BC as its Simson line?

Theorem. If the perpendicular PB_1 from a pedal point P to the side AC of ΔABC is extended to meet the circumcircle at point B_2, then BB_2 is parallel to the Simson line, C_1B_1.

Proof. Consider the figure on the right. By the inscribed angle theorem, $\angle BB_2P = \angle BCP = \angle PB_1A_1$, so that C_1B_1 and BB_2 make equal angles with PB_2 and are therefore parallel.



Theorem. The angle between the Simson lines of the two points, P and P' on the circumcircle of ΔABC is half the angular measure of the arc PP'.



Proof. Consider the figure. Note that the arcs PP' and $B_2B'_2$ are equal because PB_2 and $P'B'_2$ are parallel. Using the above theorem, we conclude that the angle between the Simson lines of points, P and P' equals the angle $\angle B_2BB'_2$ and therefore half the angular measure of the arc PP'.

Exercise. Are there points that lie on their own Simson lines? What are these points?

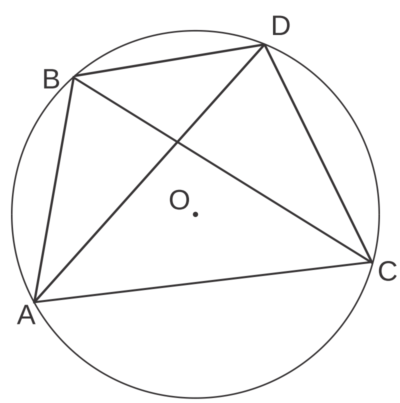
Exercise. Prove the following theorem.

Theorem. The Simson line of a points, P on the circumcircle of ΔABC bisects the segment joining that point to the orthocenter of ΔABC.

<u>**Ptolemy's theorem**</u>

Theorem. If a quadrilateral $ABCD$ is inscribed in a circle, the sum of the products of the two pairs of opposite sides is equal to the product of diagonals,

$$|AB| \cdot |CD| + |BC| \cdot |AD| = |AC| \cdot |BD|$$



Proof. Apply theorem expressing the length of the sides if a pedal triangle in the degenerate case when pedal triangle is a Simson line.

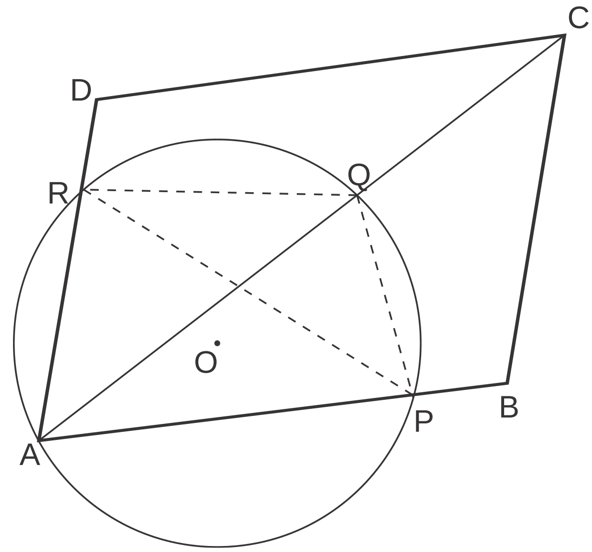
Theorem. If $ABCD$ is not an inscribed quadrilateral, the sum of the products of the two pairs of opposite sides is larger than the product of diagonals,

$$|AB| \cdot |CD| + |BC| \cdot |AD| > |AC| \cdot |BD|$$

Proof. Apply theorem expressing the length of the sides if a pedal triangle in the case when pedal triangle is not a Simson line and then use a triangle inequality.

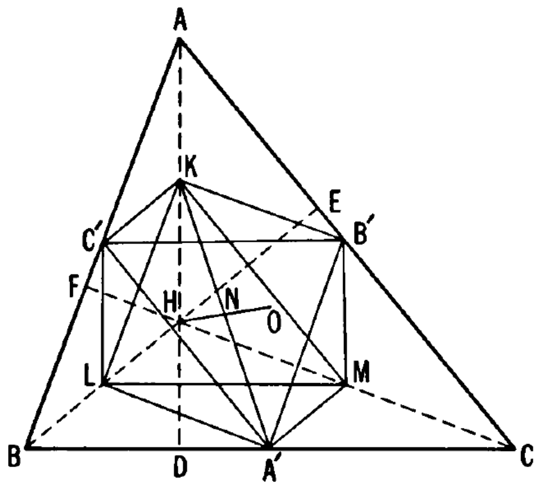
Exercise. Prove that if a circle cuts two sides and a diagonal of a parallelogram ABCD at points P, Q, R, as shown in the figure, then,

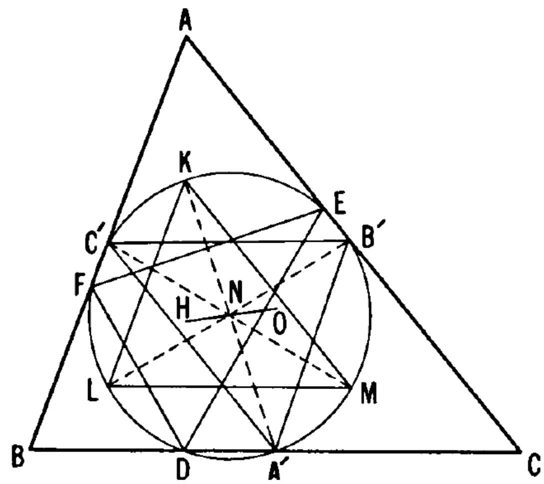
$$|AP| \cdot |AB| + |AR| \cdot |AD| = |AQ| \cdot |AC|$$



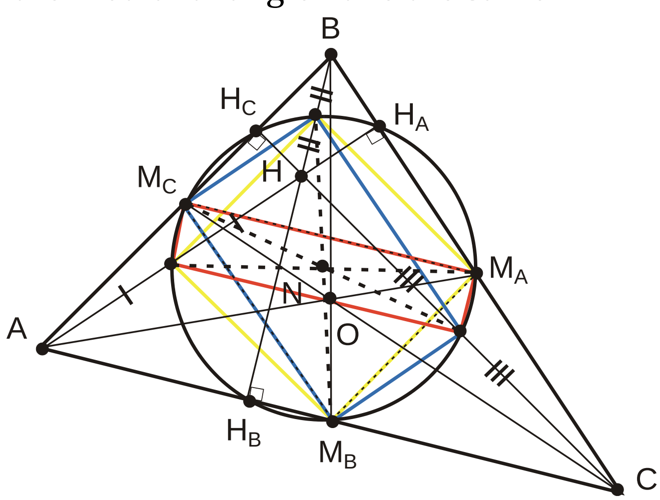
The nine-points circle problem.

Theorem. The feet of the three altitudes of any triangle, the midpoints of the three sides, and the midpoints of the segments from the three vertices lo the orthocenter, all lie on the same circle, of radius ½R.



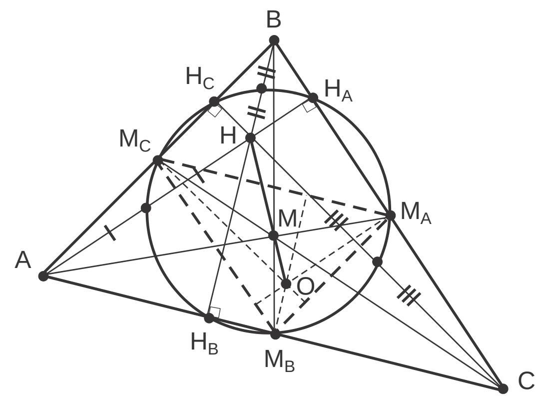


This theorem is usually credited to a German geometer Karl Wilhelm von Feuerbach, who actually rediscovered the theorem. The first complete proof appears to be that of Jean-Victor Poncelet, published in 1821, and Charles Brianson also claimed proving the same theorem prior to Feuerbach. The theorem also sometimes mistakenly attributed to Euler, who proved, as early as 1765, that the orthic triangle and the medial triangle have the same circumcircle, which is why this circle is sometimes called "the Euler circle". Feuerbach rediscovered Euler's partial result even later, and added a further property which is so remarkable that it has induced many authors to call the nine-point circle "the Feuerbach circle".



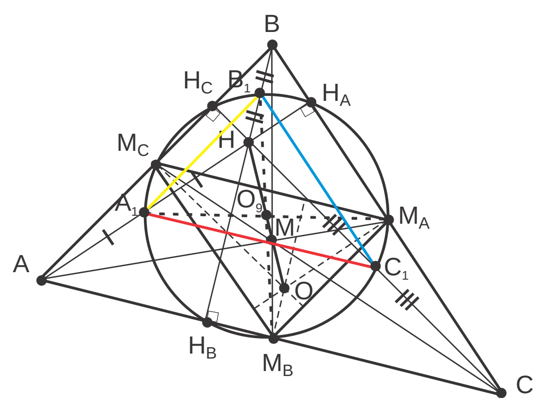
Proof. Consider rectangles formed by the mid-lines of triangle ABC and of triangles ABH, BCH and ACH.

Theorem. The orthocenter, H, centroid, M, and the circumcenter, O, of any triangle are collinear: all these three points lie on the same line, OH, which is called the Euler line of the triangle. The orthocenter divides the distance from the centroid to the circumcenter in 2: 1 ratio.



Proof. Note that the altitudes of the medial triangle$M_A M_B M_C$ are the perpendicular bisectors of the triangle ABC, so the orthocenter of $\Delta\ M_A M_B M_C$ is the circumcenter, O, of $\Delta\ ABC$. Now, using the property that centroid divides medians of a triangle in a 2:1 ratio, we note that triangles BMH and $M_B MO$ are similar, and homothetic with respect to point M, with the homothety coefficient 2.

Theorem. The center of the nine-point-circle lies on the (Euler's) line passing through orthocenter, centroid, and circumcenter, midway between the orthocenter and the circumcenter.

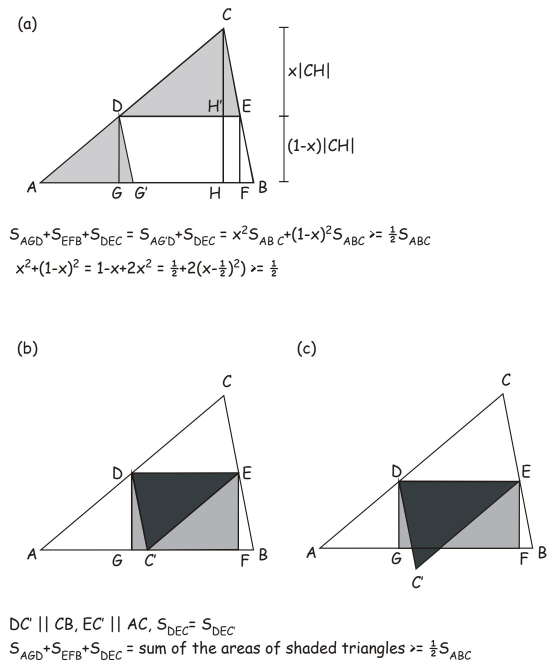


Proof. Consider the figure. Note the colored triangle $A_1 B_1 C_1$, which is formed by medians of triangles ABH, BHC and CHA, and is therefore congruent to the medial triangle $M_A M_B M_C$, but rotated 180 degrees. The 9 points circle is the circumcircle for both triangles, which means that rotation by 180 degrees about the center O_9 of the 9 point circle moves $\Delta\ M_A M_B M_C$ onto $\Delta\ A_1 B_1 C_1$, and the orthocenter, O, of the $\Delta\ M_A M_B M_C$ onto the orthocenter, H, of the $\Delta\ A_1 B_1 C_1$.

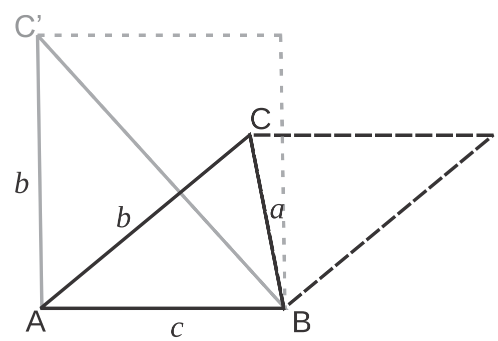
More problems.

Problem. Rectangle DEFG is inscribed in triangle ABC such that the side DE belongs to the base AB of the triangle, while points F and G belong to sides BC abd CA, respectively. What is the largest area of rectangle DEFG?

Solution. Notice similar triangles, $CDE \sim ABC$, wherefrom the vertical side of the rectangle is, $|DG| = |EF| = |CH| - |CH'| = \left(1 - \frac{|DE|}{|AB|}\right)|CH|$, so that the area of the rectangle is, $S_{DEFG} = |DE||DG| = |DE|\left(1 - \frac{|DE|}{|AB|}\right)|CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)|AB||CH| = \frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right)2S_{ABC}$. Using the geometric-arithmetic mean inequality, $\frac{|DE|}{|AB|}\left(1 - \frac{|DE|}{|AB|}\right) \leq \left(\frac{\frac{|DE|}{|AB|}+1-\frac{|DE|}{|AB|}}{2}\right)^2 = \frac{1}{4}$, where the largest value of the left side is achieved when $\frac{|DE|}{|AB|} = 1 - \frac{|DE|}{|AB|}$, and therefore $S_{DEFG} = \frac{1}{2}S_{ABC}$. There are a number of other possible solutions, some of which are shown in the figures.

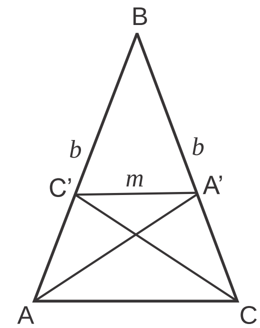


Problem. Prove that for any triangle ABC with sides a, b and c, the area, $S \leq \frac{1}{4}(b^2 + c^2)$.



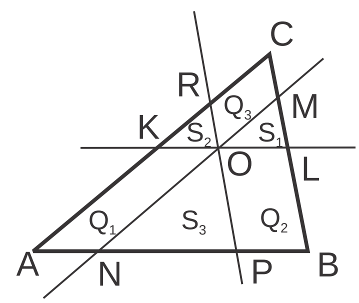
Solution. Notice that of all triangles with given two sides, b and c, the largest area has triangle ABC', where the sides with the given lengths, $|AB| = c$ and $|AC| = b$ form a right angle, $\widehat{BAC} = 90°$ (b is the largest possible altitude to side c). Therefore, $\forall \Delta ABC, S_{ABC} \leq S_{ABC'} = \frac{1}{2}bc \leq \frac{1}{2}\frac{b^2+c^2}{2}$, where the last inequality follows from the arithmetic-geometric mean inequality, $bc \leq \frac{b^2+c^2}{2}$ (or, alternatively, follows from $b^2 + c^2 - 2bc = (b-c)^2 \geq 0$.

Problem. In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).



Solution. From Thales proportionality theorem we have, $\frac{|AC|}{m} = \frac{|BC|}{|BA'|} = \frac{|BA'|+|A'C|}{|BA'|} = 1 + \frac{|A'C|}{|BA'|} = 1 + \frac{|AC|}{b}$, where we have used the property of the bisector, $\frac{|A'C|}{|BA'|} = \frac{|AC|}{|AB|} = \frac{|AC|}{b}$. We thus obtain, $|AC| = \frac{1}{\frac{1}{m}-\frac{1}{b}} = \frac{bm}{b-m}$.

Problem. Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$ (see Figure).



Solution. Denote $\frac{S_1}{S} = k_1, \frac{S_2}{S} = k_2, \frac{S_3}{S} = k_3$. Then, $\frac{S_1+S_2+Q_3}{S} = k_1 + k_2 + \frac{Q_3}{S} = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$, so, $Q_3 = 2S\sqrt{k_1 k_2} = \sqrt{S_1 S_2}$, $Q_2 = \sqrt{S_3 S_1}$, $Q_1 = \sqrt{S_2 S_3}$.