BEYOND INFINITY 5: LARGE NUMBERS AND BIG THEOREMS

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1. Homework

1. In this problem you will prove some details about countability: first, that the countable union of countable sets is countable, and second, that the finite power of countable sets is countable. Then, you can use these to prove that ε_0 is countable. Lastly, you will prove a useful fact about countable limit ordinals.

(a) Let x be a countable set, whose elements are all countable. Prove that $\bigcup x$ is countable.

(b) Let x be a countable set, and let $n \in \omega$. Prove that x^n is countable. (To define x^n, you can think of it as n copies of x in Cartesian product like $x \times ... \times x$, or you can think of it using the definition of set exponential that I gave in the first sheet.)

(c) Prove that ε_0 is countable.

(d) Prove that, given any countable limit ordinal α, there is a subset $S \subset \alpha$ whose order type is ω and whose union is $\bigcup S = \alpha$.

2. This problem is about $\mathbb{R}$ and cardinality.

(a) Let I be the unit interval, defined as all positive real numbers whose integer part is 0. So, I contains numbers of the form 0.1, 0.00101, 0.01001000100001..., etc. Prove that $|I| = |\mathbb{R}|$. Do this by proving that a bijection function exists between the two sets.

(b) Prove that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. Do this by proving that a bijection function exists between the two sets.

(c) Prove that $|\mathbb{R}| = \mathfrak{c}$. This is where the name *continuum* comes from: the real numbers are the first number line that we have that is perfectly smoothly continuous (the natural numbers and the integers occur in steps, not a continuous line - and the rational numbers appear continuous, but in fact there are a lot of missing 'holes', for example $\sqrt{2}$ - I won't define this precisely, but each irrational number is a 'hole', and the real numbers fills them in). Continuum is the cardinality of the first continuous number line.

(d) Prove that $|2^\omega| = \mathfrak{c}$, where 2^ω is defined using the definition of set exponential that I gave in the first material sheet.

3. The successor-limit strategy of Transfinite Induction. This problem is about a specific strategy of transfinite induction that I have mentioned: we create a large object by climbing ordinals, but our process for climbing is different depending on whether the target ordinal is a successor or a limit ordinal.

(a) Define a poset as follows: take the set $\mathcal{P}(\omega)$, and order these subsets of ω by proper-subset-ness. So, $x < y$ if x is a proper subset of y. Let's call this poset $\mathbb{P}$. It is sometimes referred to as the poset of subsets of ω under inclusion. Let $F \subset \mathbb{P}$ be the set of all finite subsets of $\mathcal{P}(\omega)$. Prove that, for every countable ordinal α, there is a subset of $\mathbb{P}$ of order type α.

(b) Let $T \subset \mathbb{P}$ be defined inductively as follows: for each ordinal $\alpha \in \omega^2$, if α is a successor ordinal, then add to T all subsets of ω that are one element bigger than a set already in ω. That is, for all $x \in T$ and for all $n \in \omega$, let $x \cup \{n\}$ be in T. For limit ordinals α, if $\alpha = 0$ then $T = \{\emptyset\}$, and if $\alpha > 0$, then let $\beta \in \alpha$ be the biggest limit ordinal less than α, let m be the smallest natural number that is not in the set that was added to T at step β, let $Z(m) = m \cdot p$ where p is the smallest prime number not in the prime factorization of m, and then add to T the set $\omega \setminus \{k \cdot Z(m) \big| k \in \omega\}$; so, at the α step, we add only one set to T, and that is the set of all natural numbers that are *not* a multiple of $Z(m)$. What's the tallest well-ordered subset of T? ("Tallest" means its order type is a bigger ordinal than the order type of any other well-ordered subset of T.)

4. A closer look into ω_1.

(a) Define a poset as follows: let $\mathbb{O}$ be the set of partial functions whose domain is ω and whose range is an ordinal; let the order relation be the one described in the Aronszajn tree section of the material sheet. Prove that this poset is a tree, and prove that it has no uncountable branches.

(b) Describe the ω-level of $\mathbb{O}$, and prove that it is uncountable.

(c) What is the height of $\mathbb{O}$?

5. Aronszajn!

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(a) Define a function g on ω as follows: for each natural number n, write n as $2^k \cdot r$ for an odd number r, and let $g(n) = k \cdot \omega + (r-1)/2$. Prove that the range of g is an ordinal, and thus g is in $\mathbb{O}$. What is the range of g? Which level of $\mathbb{O}$ is it in?

(b) Define a function h as follows: for each natural number $n = 2^k \cdot r$ for odd r, let $h(n) = g(n)$ if $r > 1$ and $h(n) = g(\sqrt{n})$ if $r = 1$. For which n is h defined? (If $\sqrt{n}$ is irrational, then h is undefined.) Prove that h is a coinfinite element of $\mathbb{O}$ in the same level as g.

(c) Let $A = O(h) \subset \mathbb{O}$. Let $AR = \{f \in \mathbb{O} \big| g \in A \wedge f =^{\mathbb{O}} g\}$, where $=^{\mathbb{O}}$ means that f and g are finite-equivalent and also in the same level of $\mathbb{O}$. Prove that AR is a tree, and that its levels are all countable. Then prove that, for each ordinal α that is less than the range of g, the α-level of AR is a subset of the α-level of $\mathbb{O}$ (thus, the levels correspond).

(d) Understand and generalize the concepts in this problem to complete AR into a full Aronszajn tree.

2. ε_0

1. Let c_α be the α^{th} ordinal that satisfies the equation $x \cdot \omega = x$. So, $c_0 = 0$ is considered the 0th ordinal to satisfy this equation since $0 \cdot \omega = 0$; then, c_1 is the smallest ordinal after 0 to satisfy this equation, which is ω^ω. Then c_1 is the smallest ordinal larger than c_1 to satisfy the equation, etc.

(a) Prove that $c_1 = \omega^\omega$. To do this, you must prove both that ω^ω satisfies the equation, and that no smaller ordinal other than 0 does.

(b) Determine and describe c_2.

(c) Determine and describe c_ω.

(d) Prove that $c_{\omega_1} = \omega_1$.

(e) Is there any countable ordinal γ such that $c_\gamma = \gamma$?

2. This problem concerns ordinal exponentiation with ω involved.

(a) Prove that, given any ordinal $\alpha > 1$, we have $\alpha^\omega > \alpha$.

(b) Prove that $\omega^{\omega_1} = \omega_1$.

(c) Is there a countable ordinal γ such that $\omega^\gamma = \gamma$?