BEYOND INFINITY 4:

JADE NINE

1. Practice Problems

1. Prove that $\omega + \omega = 2 \cdot \omega$.
2. Prove that $\omega \cdot \omega$ is a limit ordinal.
3. Calculate $|\mathcal{P}(3)|$.
4. Is there a biggest ordinal?

2. Practice Problem Solutions

1. By definition, $\omega + \omega = \{0\} \times \omega \cup \{1\} \times \omega$ under lexicographic ordering (and converted to an ordinal); also by definition, $2 \cdot \omega = 2 \times \omega = \{0,1\} \times \omega$ under lexicographic ordering (and converted to an ordinal). But $\{0,1\} \times \omega$ is the set of all ordered pairs of the form $(0,t)$ or $(1,t)$ for $t \in \omega$ - this is the same as taking the union of the set of all ordered pairs of the form $(0,t)$ and all ordered pairs of the form $(1,t)$. Essentially, $\{0\} \times \omega \cup \{1\} \times \omega = \{0,1\} \times \omega$. Any ordered pair in one set must also be in the other set, thus by axiom of extensionality, the sets are equal.

2. We need to prove that $\bigcup \omega \cdot \omega = \omega \cdot \omega$. We can prove this by showing that each set is a subset of the other. Firstly, $\bigcup \omega \cdot \omega \subset \omega \cdot \omega$: suppose $t \in \bigcup \omega \cdot \omega$, then by definition of union, $\exists r \in \omega \cdot \omega (t \in r)$, but because $\omega \cdot \omega$ is an ordinal, $r \in \omega \cdot \omega \implies r \subset \omega \cdot \omega$, thus $t \in r \implies t \in \omega \cdot \omega$. Secondly, $\omega \cdot \omega \subset \bigcup \omega \cdot \omega$: suppose $t \in \omega \cdot \omega$, then - as discussed in an example in the materials sheet - $\exists n \in \omega (t \in n \cdot \omega)$, but $n \cdot \omega$ is an element of $\omega \cdot \omega$ for all n, thus $t \in n \cdot \omega \implies t \in \bigcup \omega \cdot \omega$.

3. $|\mathcal{P}(3)| = 8$. Any subset of 3 involves 3 choices of elements: for each of 0, 1, 2, you decide that the element is in the subset or it is not; therefore, there are $2 \cdot 2 \cdot 2$ total possibilities for picking a subset, and $2 \cdot 2 \cdot 2 = 8$.

4. No; suppose α is an ordinal, then the successor of α, given by $\alpha + 1 = \alpha \cup \{\alpha\}$, is an ordinal bigger than α. (This is similar to the proof that there is no biggest finite number: given any number, you can add 1 to get a bigger number.)

3. Homework Problems

1. This problem concerns noticing some facts about ordinal arithmetic.
  (a) Prove that for any ordinals α, β, which are both larger than 0, $\alpha + \beta$ can never be equal to α.
  (b) Find and describe a pair of ordinals α, β larger than 0 such that $\alpha + \beta = \beta$.
  (c) Prove that for any ordinals α, β, which are both larger than 1, $\alpha \cdot \beta$ can never be equal to β.
  (d) Find and describe a pair of ordinals α, β larger than 1 such that $\alpha \cdot \beta = \alpha$.

2. This problem is about successor and limit ordinals.
  (a) What are the successor ordinals in ω?
  (b) What are the limit ordinals in $\omega \cdot \omega$?

3. This problem concerns infinite ordinals.
  (a) Is it possible to find a pair of ordinals, α, β, both infinite, such that $\alpha + \beta = \beta$?
  (b) Prove that any infinite ordinal α can be order-embedded into a proper subset of itself.

4. This problem concerns cardinality of infinite ordinals.
  (a) Prove that, for any infinite ordinal α, $|\alpha + 1| = |\alpha|$.
  (b) Prove that $|\omega \cdot \omega| = |\omega|$.

4. Bonus Problems

1. What ordinal is $\omega^{[2]}$ with the triangle order $<_\triangle$ isomorphic to?
2. Prove that, given an ordinal α such that $\forall t \in \alpha (t + 1 \in \alpha)$, that α is a limit ordinal.
3. Is it possible to find a pair of ordinals, α, β, both infinite, such that $\alpha \cdot \beta = \alpha$?
4. Given an ordinal α, let $L(\alpha)$ be the subset of all limit elements of α, interpreted as a poset (using the same order that α has). Find and describe an ordinal α such that $L(\alpha)$ is isomorphic to α.

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