# BEYOND INFINITY 4: ORDINALS AND CARDINALS

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## Ordinals

Recall the definition of ordinals.

**Definition 1** (Ordinals)**.** An ordinal is a set $\alpha$ that satisfies the following properties:

- $\alpha$ with the relation $\in$ is a well-ordered poset.
- $\forall \gamma \in \alpha (\gamma \subset \alpha)$.

The second property is called transitivity, because it is equivalent to $\forall \gamma \forall \delta (\delta \in \gamma \land \gamma \in \alpha \implies \delta \in \alpha)$. (This is a useful property that I did forgot to mention explicitly last time in the definition, but it is important for ordinals.)

Additionally, here is a useful fact about ordinals.

**Theorem 1** (Ordinal Equality)**.** *If two ordinals are isomorphic as posets, then they are equal as sets.*

Thus, ordinals are, in a sense, the ideal representatives of well-ordered sets. As you will see below, any well-ordered set is isomorphic to some ordinal; moreover, from the theorem above, any two isomorphic ordinals are equal, thus each ordinal is unique in its shape as a poset. We can think of ordinals as the *order type* of a well-ordered set: a well-ordered set of five elements has order type 5; a well-ordered set that has infinitely many elements starting at a base and going up one by one will have order type $\omega$; etc.

## 1. Correspondence Theorem

**Theorem 2** (Correspondence Theorem)**.** *Given any well-ordered poset $W$, there is an ordinal $\gamma$ that is isomorphic to $W$.*

Given a well-ordered poset $W$, you can construct an ordinal by building things up set-wise: start with the smallest element of $W$ and let that correspond to the empty set. Then let the second smallest element correspond to 1, the next smallest after that correspond to 2, the next to 3, and so forth. Generally speaking, given any element $w \in W$, let $g(w)$ be defined as $\{r \in W \big| r < w\}$. Then let $\gamma = \{g(w) \big| w \in W\}$. $\gamma$ will be an ordinal that encodes the exact shape of $W$, except that it 'thinks' that all its elements are ordinal sets and that its order relation is $\in$, not whatever elements and relation $W$ might have had. Thus, $\gamma$ will end up being an ordinal isomorphic to $W$.

## 2. Successor and Limit Ordinals

**Definition 2** (Successor Ordinal)**.** Given any ordinal $\alpha$, the ordinal $\alpha + 1 = \alpha \cup \{\alpha\}$ is the successor of $\alpha$. If an ordinal is a successor of another ordinal, it is called a successor ordinal.

**Definition 3** (Limit Ordinal)**.** Given any ordinal $\alpha$, if $\alpha = \bigcup \alpha$, then $\alpha$ is called a limit ordinal. If an ordinal is a union of an infinite increasing sequence of ordinals, then it is a limit ordinal; the empty set, or 0, is also a limit ordinal.

In line with our definition of successor and limit elements of posets, successor ordinals are successor elements of all ordinals bigger than them; limit ordinals are limit elements of all ordinals bigger than them. I'll leave this to you to interpret and make the connection.

Some examples might be helpful, though. Draw out a dot-and-line diagram of the ordinal $\omega + \omega$ as a poset - recall that this is $\omega$, plus another copy of $\omega$ on top of it. The limit ordinals of $\omega + \omega$ are 0 and $\omega$. The successor ordinals are 1, 2, 3, 4, ..., and $\omega + 1$, $\omega + 2$, $\omega + 3$, $\omega + 4$, ....

## 3. Ordinal Arithmetic

If you haven't read it yet, read through the Batch 5 posets from last week.

In any case, the point here is to try to put ordinals together into bigger ordinals. We have addition and multiplication, both of which are noncommutative.

The idea for addition is to put one ordinal on top of the other. So, $\omega + 2$ is what you get by putting 2 'on top' of $\omega$, essentially placing 2 elements that are greater than everything in $\omega$. This is formalized as follows.

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**Definition 4** (Addition)**.** Given ordinals $\alpha$ and $\beta$, we have $\alpha+\beta$ is the ordinal isomorphic to the poset $(\{0\}\times\alpha)\cup(\{1\}\times\beta)$ under lexicographic ordering.

Then there is multiplication. The idea here is, given $\alpha$ and $\beta$, to put $\alpha$ many copies of $\beta$ on top of each other. Equivalently, draw out $\alpha$ and replace each element with a full copy of $\beta$. This is formalized as follows.

**Definition 5** (Multiplication)**.** Given ordinals $\alpha$ and $\beta$, we have $\alpha \cdot \beta$ is the ordinal isomorphic to the poset $\alpha \times \beta$ under lexicographic ordering.

Try to convince yourself that $\omega$, $\omega + 2$, and $2 \cdot \omega$ are all different ordinals, i.e. they are not isomorphic. Then, try to see why $2 + \omega$ and $\omega \cdot 2$ are both isomorphic to $\omega$.

Additionally, here is a useful example: suppose we take $1 \cdot \omega, 2 \cdot \omega, 3 \cdot \omega, 4 \cdot \omega, ...$ and union all these ordinals together. The result will be $\omega \cdot \omega$. The shape of this ordinal is an infinitely increasing tower of copies of $\omega$. Intuitively, any element from $\omega \cdot \omega$ corresponds to an ordered pair $(a, b)$ where $a, b \in \omega$, and this pair will correspond to $a \cdot \omega + b$, which is an element of $(a + 1) \cdot \omega$.

## 4. Cardinality

Infinity is special in a peculiar way, take a look at this:

Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = 2n$.

Then $f$ is a function from $\mathbb{N}$ into itself, its domain is all of $\mathbb{N}$, but its range is not all of $\mathbb{N}$. And yet, it manages to be an order-embedding, so it is possible to order-embed $\mathbb{N}$ into a proper subset of itself. This is not possible for any finite well-ordered poset. Yet, it is possible for every infinite well-ordered poset.

It turns out, functions are useful for comparing and determining sizes of sets. We discussed last week that, given any finite numbers $a$ and $b$, there is no invertible function from $a$ to $b$ unless $a = b$. In fact, two finite sets have the same number of elements if and only if there is an invertible function between the two sets. Such an invertible function is said to be a 1:1 correspondence: you think of it as 'corresponding' all elements of $a$ to those of $b$. Such a correspondence is also sometimes called a bijection.

The *size* of a set will be formalized by a notion that I'll call cardinality. Formally, like this:

**Definition 6** (Same Cardinality)**.** Two sets $x$, $y$ are said to have the same cardinality if there is a 1:1 correspondence function from $x$ to $y$.

The cardinality of a set is usually written with absolute value symbols, for example $|x|$ means the cardinality of $x$; this is useful for comparing cardinalities, because we can just say $|x| = |y|$ instead of "$x$ has the same cardinality as $y$". Formally, $|x|$ is defined as the smallest ordinal that has the same cardinality as $x$: therefore, $|x|$ is, by definition, a set (and in fact, it is an ordinal). For example, $|\{2, 3, 4\}| = 3$.

**Definition 7** (Cardinality)**.** Given a set $x$, the cardinality of $x$ is the smallest ordinal $\alpha$ such that $|x| = |\alpha|$.

**Theorem 3** (Finite Cardinality)**.** *No two elements of $\omega$ have the same cardinality. Furthermore, the cardinality of every finite set is an element of $\omega$.*

Thus, $\omega$ carries inside it all possible finite cardinalities. And, for any $n \in \omega$, $|n + 1| \neq n$.

However, $|\omega + 1| = \omega$. To see this correspondence, take the 1 at the top of $\omega + 1$ and move it to the bottom, to get $1 + \omega$. In other words, use the function $f : \omega + 1 \to \omega$ given by $f(\omega) = 0$ and $f(n) = n + 1$ (recall that $\omega + 1$ is the set $\{0, 1, 2, ..., \omega\}$). A similar concept is found in the famous Hotel Infinity analogy, which you can read about if you haven't heard of it. Another way to think about it is that an ultimately-lazy machine can procrastinate a task infinitely by doing every other task first, putting off that task until infinite time into the future - this is the $\omega + 1$ strategy; in comparison, an efficient machine would do that task first, and then get to all the other tasks. Since the set of tasks that the two machines complete is the same, this sets up the desired 1:1 correspondence of the two sets.

However you want to think about it, the result is that $\omega$ is the cardinality of more than one ordinal: $|\omega| = \omega$, but $|\omega+1| = \omega$, also you can prove that $|\omega+2| = \omega$, $|2\cdot\omega| = \omega$, $|n\cdot\omega| = \omega$, in fact it is even possible to prove that $|\omega\cdot\omega| = \omega$.

So, how far does $\omega$ reach? What is the biggest ordinal whose cardinality is $\omega$? *Is* there a biggest ordinal whose cardinality is $\omega$? Does *every* infinite ordinal have cardinality $\omega$?

In the vein of the latter thought, here are two famous results about cardinality:

**Theorem 4** (Integer Cardinality)**.** $|\mathbb{Z}| = \omega$

*Proof.* To prove this, I will fit the integers into $\omega$ by fitting the positive integers into the even natural numbers, and the negative integers into the odd natural numbers. Let $f : \mathbb{Z} \to \omega$ be given by $f(x) = 2x$ for $x \geq 0$, and $f(x) = -2x - 1$ for $x < 0$. Then $f$ is invertible, you can check that the inverse is $f(n) = n/2$ for even $n$ and $f(n) = -(n+1)/2$ for odd $n$. □

**Theorem 5** (Rational Cardinality)**.** $|\mathbb{Q}| = \omega$

*Proof.* This is perhaps the more surprising one, but nonetheless it is true. To prove it, I'll use the fact that the rational numbers are quotients of integers, i.e. every rational number is $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ (and $q \neq 0$). This way of looking at the rationals is more cardinality-intuitive than imagining the rationals as a poset or a number line. If we think of rationals as quotients of integers, then we can plot them on a coordinate plane, where $\frac{p}{q}$ is plotted at the point $(p, q)$.

Now, to make a 1:1 correspondence from the rationals to $\omega$, it's enough for me to pick out the rational numbers one by one and map those sequentially to 0, 1, 2, 3, etc., in $\omega$. The end result will be a correspondence from rational numbers to elements of $\omega$, and as long as I cover all the rational numbers, it will be a perfect 1:1 correspondence.

To cover all rational numbers, use a diagonal argument. Start by mapping $(0, 1) \in \mathbb{Q}$ to $0 \in \omega$. Then, draw a diagonal line through all rational numbers whose numerator (in absolute value) and denominator (in absolute value) sum to 1 (it'll be two diagonal lines, one for the positives and one for the negatives). Try drawing this out in a picture if you have some square ruled paper and can plot the rational numbers on the grid. Make sure to skip over all ordered pairs that don't correspond to a legitimate rational number in reduced form (for example, $(2, 0)$ or $(2, 4)$). Then make another pair of diagonal lines through the rationals whose numerator and denominator sum to 3. By this process, you can count out the rationals one by one, mapping them each to elements of $\omega$. Continue until you go through all values of $k$, for rational numbers whose numerator and denominator (in absolute value) sum to $k$, which will cover all rational numbers.

Essentially, $f : \mathbb{Q} \to \omega$ is given by $f(\frac{0}{1}) = 0$, $f(\frac{1}{1}) = 1$, $f(\frac{-1}{1}) = 2$, $f(\frac{2}{1}) = 3$, $f(\frac{1}{2}) = 4$, $f(\frac{-2}{1}) = 5$, $f(\frac{-1}{2}) = 6$, $f(\frac{3}{1}) = 7$, $f(\frac{1}{3}) = 8$, $f(\frac{-3}{1}) = 9$, $f(\frac{-1}{3}) = 10$, $f(\frac{4}{1}) = 11$, $f(\frac{3}{2}) = 12$, $f(\frac{2}{3}) = 13$, $f(\frac{1}{4}) = 14$, $f(\frac{-4}{1}) = 15$, and so on. □

In fact, the cardinality $\omega$ is so powerful that our axioms at present are not strong enough to construct an infinite set that has a cardinality that is not $\omega$. For this, we will need one more axiom. But before that, we shall invent a word: we'll say that a set is countable if its cardinality is $\omega$. Intuitively, a 1:1 correspondence with $\omega$ is a 'counting' of a set: there is an element that maps to 0, an element that maps to 1, an element that maps to 2, an element than maps to 3, an element that maps to 4, etc.

## 5. Power Set

**Axiom 6** (Powerset)**.** *For any set $x$, there exists a set of all subsets of $x$, called the powerset of $x$, written $\mathcal{P}(x)$. An element of $\mathcal{P}(x)$ is a subset of $x$.*

And with it, this wonderful but calamitous theorem:

**Theorem 6** (UNCOUNTABLE!)**.** *$|\mathcal{P}(\omega)| \neq \omega$.*

It's calamitous - in a good way - because it is the root of many famous results about logic and theory. We will talk about these next week, in detail about some of them, and passingly about others. Some are a little counterintuitive, but I'll let you judge for yourself when we get there.

Anyways, the cardinality of $\mathcal{P}(x)$ is called continuum, written $\mathfrak{c}$. It is an ordinal, and it is a very big one.

## 6. △

Let me define an ordering on subsets of natural numbers. Let's say that, given $A \subset \omega$ and $B \subset \omega$, that $A < B$ if $A$ has an element that's smaller than everything in $B$. For example, $\{1, 2\} < \{2, 3\} < \{5, 9\}$. How do we compare $\{1, 2\}$ and $\{1, 3\}$? The idea is to remove everything that the sets have in common. Let's write this process with $\triangle$ for convenience, so $A \triangle B = (A \cup B) \setminus (A \cap B)$. So, $\{1, 2\} \triangle \{1, 3\} = \{2, 3\}$. This lets us write a formal, legitimate procedure for comparing sets of the same size: $A < B$ iff $min(A \triangle B) \in A$. In cases where $A$ and $B$ might be of different sizes, we might have $A \subset B$, so we say $A < B$ if $A \subset B$ or, if $A$ is not a subset of $B$, use the $\triangle$ method.

The symbol $\triangle$ is read as "symmetric difference".

Next, let me write $\omega^{[n]}$ to denote the set of all $n$-element subsets of $\omega$. That is, all subsets of $\omega$ that have cardinality $n$ are elements of $\omega^{[n]}$. Similarly, you can guess what $\omega^{[<\omega]}$ should mean.

**Definition 8** (Symmetric Difference)**.** Given sets $x$, $y$, $x \triangle y = (x \cup y) \setminus (x \cap y)$.

**Definition 9** (Triangle Order)**.** Given two subsets $a, b$ of a well-ordered poset $P$, the triangle order $<_\triangle$ is the order relation defined as $a < b \iff (a \subset b) \lor \big((a \not\subset b) \land (min_P(a \triangle b) \in a)\big)$, where $min_P$ is determined using the order relation of $P$.

**Definition 10** (Bracket Exponential)**.** Given a set $x$, $x^{[n]} = \{t \in \mathcal{P}(x) \big| |t| = n\}$.

**Definition 11** (Finitary Exponential)**.** Given a set $x$, $x^{[<\omega]} = \{t \in \mathcal{P}(x) \big| |t| \in \omega\}$.