BEYOND INFINITY 3: HOMEWORK

JADE NINE

1. Optional Practice

Do these problems if you want some exercises to help you understand the concepts of this week. The answers are given below.

1. Axiom Practice
(a) Use the axiom of extensionality to prove that $\{1, 2, 3, 3\} = \{1, 2, 3\}$. This justifies the idea that sets ignore repeat elements.
(b) Explain why, for any x, $\{x, x\} = \{x\}$. Then explain how to use the axiom of pairing to construct the set $\{x\}$.
(c) Use the axiom of extensionality to prove that, given any two sets x and y, if x and y both have no elements in them, then $x = y$. This justifies the idea that the empty set is unique: all sets with no elements in them are equal to each other.
(d) Describe the set of even natural numbers using set-builder notation.
(e) Given some nonempty set x, let $y = \{t \in x | t \neq t\}$. What sort of set is y?
(f) Given two sets x, y, use the axioms of pairing and union to construct the set $x \cup y$.
(g) Given two sets x, y, which axioms must be used in order to construct the set $x \cap y$?

2. Well-ordered Sets
(a) Draw a dot-and-line diagram for a well-ordered poset on 5 elements.
(b) Prove that any two well-ordered posets on 5 elements are isomorphic.
(c) Prove that, for any finite natural number n, any two well-ordered posets on n elements are isomorphic.

2. Optional Practice Solutions

1. Axiom Practice
(a) Given some arbitrary set a, we have to prove that $a \in \{1, 2, 3, 3\} \iff a \in \{1, 2, 3\}$. To prove this, I will prove that $a \in \{1, 2, 3, 3\} \implies a \in \{1, 2, 3\}$ and $a \in \{1, 2, 3\} \implies a \in \{1, 2, 3, 3\}$. So, if $a \in \{1, 2, 3, 3\}$, then a is 1, 2, or 3. All of these elements are in $\{1, 2, 3\}$, therefore $a \in \{1, 2, 3\}$. To prove the opposite, if $a \in \{1, 2, 3\}$ then a is 1, 2, or 3; all of these elements are in $\{1, 2, 3, 3\}$, therefore $a \in \{1, 2, 3, 3\}$.
(b) If $a \in \{x, x\}$, then $(a = x) \vee (a = x)$, therefore $a = x$. If $a = x$, then $a \in \{x\}$. Therefore, if $a \in \{x, x\}$, then $a \in \{x\}$. Proving the opposite is similar, so we can deduce that $a \in \{x\} \implies a \in \{x, x\}$. Thus $\{x, x\} = \{x\}$ by extensionality.
(c) $a \in x \implies a \in y$ is true because $a \in x$ is false (x has no elements), therefore the implication is true (remember that a logical implication is considered logically true if the premise is false). Similarly, $a \in y \implies a \in x$. Thus, $a \in x \iff a \in y$, therefore $x = y$ by extensionality.
(d) $\{t \in \mathbb{N} | \exists r \in \mathbb{N}(t = 2r)\}$.
(e) y is the empty set. All its elements are not equal to themselves, which is a logical contradiction for any element, therefore no element in y can possibly exist; as a result, y has no elements in it. Such a definition is logically sound: as long as y is empty, there will be no logical contradiction.
(f) Use pairing to construct $\{x, y\}$, then use union to construct $\bigcup\{x, y\}$.
(g) You need pairing, union, and separation. Use pairing and union to construct $x \cup y$. Then use separation to construct $\{t \in (x \cup y) | t \in x \wedge t \in y\}$.

2. Well-ordered Sets
(a) You should have a straight vertical line with 5 dots on it, or something similar.
(b) By the Embedding Theorem, there is an order-embedding from one of the posets into the other. But an order-embedding must map different elements in the domain to different elements in the range, so the 5 elements of the domain must end up covering all 5 elements in the range. This produces an invertible correspondence. Thus, the posets are isomorphic.

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(c) The proof for this is basically the same: using the fact that n is finite, any order embedding from a poset on n elements must have its domain and range both have exactly n elements in them; because the range has n elements, this embedding must cover all elements in the range; one can then easily define an inverse function to prove that the embedding is invertible.

3. Common Problems

1. (a) Prove that if $x \subset y$ and $y \subset x$, then $x = y$.
(b) Prove that if $x \cup y = x \cap y$, then $x = y$.
(c) Which axioms are required in order to construct the finite ordinals n? You will have to start with the existence axiom, but then describe what you would use from there.

2. Let α and β be ordinals. Suppose that α and β are isomorphic, using the element relation $\in$ as their order relation. Prove that $\alpha = \beta$ (i.e., that α and β are equal as sets).

4. Select-A-Problem

1. (a) Explicitly describe an order relation on the set $\mathbb{Z}$ that produces a well-ordered poset.
(b) Explicitly describe an order relation on the set $\mathbb{Q}$ that produces a well-ordered poset.

2. Given an ordinal α,
(a) Prove that $\bigcup \alpha$ is an ordinal.
(b) Prove that $\alpha \cup \{\alpha\}$ is an ordinal.
(c) Prove that $\bigcup \{\alpha \cup \{\alpha\}\}$ is an ordinal.
(d) Let α^* be the reverse ordering on α, i.e. the same set of objects but everytime $p < q$ in α, we have $p > q$ in α^*. Is α^* well-ordered?

3. (a) Given well-ordered posets P and Q, prove that $P \times Q$ with the lexicographic ordering is a well-ordered poset.
(b) Prove that a totally ordered poset P is well-ordered if and only if there is no order-embedding function from $\mathbb{Z}^-$ to P.

4. (a) Describe a poset that has a proper subset that is isomorphic to the poset itself.
(b) Prove that it is impossible for a segment of a well-ordered poset to be isomorphic to the poset itself.
(c) Is it possible for a segment of a non-well-ordered poset to be isomorphic to itself?

5. An Adventure in Batch 4.
(a) Draw a dot-and-line diagram for $\mathbb{Z}V$.
(b) Prove that any two elements in $\mathbb{X}$ are compatible.
(c) Construct or describe an order-embedding v from ω to $\mathbb{Y}$ such that, for any $m, n \in \omega$, if $m < n$, then $O(v(n)) \setminus O(v(m))$ is infinite.
(d) Given two elements $a, b \in \mathbb{M}$, prove that if $O(b) \setminus O(a)$ is infinite, then there is an order embedding from $\mathbb{Z}$ to $O(b) \setminus O(a)$.