BEYOND INFINITY 3: ORDINALS & AXIOM OF CHOICE

JADE NINE

1. Omega

A reminder of definitions:

- Segment: a segment of an element x of a poset P is the set of all elements less than x, i.e. the set of all $y \in P$ such that $y < x$. Segments are typically written with an O, so $O(x)$ is the segment of x. (Segments are also called *initial segments.*)
- Successor: an element x of a well-ordered poset P is called a successor element if $O(x)$ is a bounded poset.
- Limit: an element x of a well-ordered poset P is called a limit element if $O(x)$ is not a bounded poset, or if $O(x)$ is the empty set.

Theorem 1 (Omega). *Any two well-ordered posets that are infinite and have only one limit element are isomorphic. Such a poset is called ω.*

Proof. Let P, Q be two well-ordered posets that are infinite and have only one limit element. We will construct a function $f : P \to Q$ that is an isomorphism; we'll just call this function f.

P is well-ordered, so the subset $P \subset P$ has a minimum, call it p_0. Q is well-ordered, thus by a similar argument, also has a minimum element, let's call it q_0. Let $f(p_0) = q_0$. Note that both p_0 and q_0 are limit elements, thus the rest of the elements in P and Q are all successor elements.

Now take the subset $P \setminus \{p_0\}$; it is a subset of P, thus it has a minimum element; call this element p_1. By a similar argument, $Q \setminus \{q_0\}$ has a minimum element, call it q_1. Let $f(p_1) = q_1$.

Next, take the minimum of $P \setminus \{p_0, p_1\}$ and call it p_2; take the minimum of $P \setminus \{q_0, q_1\}$ and call it q_2. Let $f(p_2) = q_2$. You might be able to guess where this is going.

The question is whether the resulting function will end up being an isomorphism. First of all this function, f, which maps p_n to q_n, is clearly an order-embedding: $p_k < p_n$ if and only if $q_k < q_n$. Is f an invertible function?

In fact, we still have to check if its domain is P, because we didn't prove that all the elements of P will eventually be in the domain of f. To see why this must be the case, suppose there are some elements in P that are not in the set $\{p_0, p_1, p_2, ...\}$, and thus not in the domain of f. Let the set of all such elements be called P'. $P' \subset P$ and P is well-ordered, thus P' has a minimum element; call it p'. We know p' is greater than every element in $\{p_0, p_1, p_2, ...\}$, because if not - say for example $p' < p_8$ - then the minimum of $\{p_0, p_1, ..., p_7\}$ is p' and not p_8, contradicting the construction of p_8. We also know that the segment of p' cannot contain any elements not in the domain of f, because p' is the smallest such element. Therefore, from these two facts, we deduce $O(p') = \{p_0, p_1, p_2, ...\}$. Notice that $\{p_0, p_1, p_2, ...\}$ is not a bounded poset; thus, by definition, p' is a limit element. This contradicts the condition that P have only one limit element, which is p_0. Therefore, the domain of f is all of P.

By a similar argument, the range of f is all of Q. It is not hard to see that f is then invertible, with $f(p_n) = q_n$ and $f^{-1}(q_n) = p_n$.

Thus f is an invertible order-embedding. Therefore P is isomorphic to Q. This completes the proof. ☐

It is standard to represent ω as $\mathbb{N}+$, i.e. $\omega = \{0, 1, 2, 3, ...\}$ under the usual order relation of numbers.

Now for the big fish.

Theorem 2 (Embedding Theorem). *Given any two well-ordered posets P and Q, there exists an order-embedding from P to Q or there exists an order-embedding from Q to P. If both order-embeddings exists (i.e. from P to Q and from Q to P), then P and Q are isomorphic.*

Proof. Suppose P and Q are well-ordered posets. We will construct an order-embedding $f : P \to Q$, by a method known as *transfinite induction.* It goes like this:

Let $D \subset P$ be the domain of f, which is a set we will add elements to. Let Similarly, R be the range of f. Let's suppose D and R are both proper subsets of P and Q, respectively. Then $P \setminus D$ and $Q \setminus R$ are nonempty subsets of P and Q, thus have minimum elements m and n, respectively. Let's add m to D, n to R, and declare $f(m) = n$. While we're at it let's also declare a function g with the opposite correspondence: R is the domain of g, D is the range of g, and $g(n) = m$.

Date: July 12, 2020.

Repeat this process as far as you can possibly go. When will it no longer be possible to grow the function f (and g) by adding more elements to its domain and range?

As long as D and R are both proper subsets of P and Q, then we can use the above process to add another element. Therefore we can only stop when $D = P$ or $R = Q$. If $D = P$, then f is a function from P to Q; if $R = Q$, then g is a function from Q to P.

Is f an order-embedding? Suppose $a, b \in P$ with $a < b$. Then, at the step that a was added to the domain of f, the element $f(a) \in Q$ was chosen as the minimum of the elements not yet in the range of f. But $a < b$, so b is not yet in the domain of f, and thus $f(b) \in Q$ is not yet in the range of f, therefore the selection $f(a)$ must be less than $f(b)$ which is not yet in the range of f. One might think that $f(b) \in Q$ could possibly be selected as $f(a)$, but this would lead to a problem: by the time we reach b to add to the domain of f, $f(b) = f(a)$ is already in the range of f, and thus cannot possibly be selected as $f(b)$ which is outside the range of f.

By a similar argument, g is an order-embedding.

If we hit $D = P$ and $R = Q$ at the same time, then f and g are inverse functions of each other, therefore P and Q are isomorphic.

This completes the proof.

$\square$

What just happened, and why would such a process be called transfinite induction?

We used a process that, at every step, looks for the smallest element not yet reached, and uses that element as the next step in the induction. Because the poset we are working on is well-ordered, the subset of stuff we have not yet reached will have a minimum element whenever it is nonempty. Thus, we can continue the induction process until we reach all elements of our poset. This process is transfinite because there might be elements that take infinitely many steps to reach. The way that we reach these elements is by declaring that it is logically impossible *not* to reach them.

To see an example of such a transfinite inductive step, I will introduce you to the poset known as $\omega + 1$, defined as follows:

Definition 1 ($\omega + 1$)**.** Let $\omega + 1$ be defined as a poset on the set $\{0, 1, 2, 3, ...\} \cup \{\omega\}$, and let the order relation be the standard order relation on numbers, plus the declaration that $n < \omega$ for all natural numbers n.

2. Axioms, Part 2

Recall that we are now working with sets of sets. In case you were wondering whether this is circular logic - it certainly seems to be, but it isn't. There is one asbolute set that doesn't require any circular definition: the empty set.

So essentially all our sets boil down to sets that contain sets that contain the empty set plus sets that contain the empty set and a set that contains a set that contains the empty set and (and so forth). The trick is to find a way to make it meaningful.

Recall the axioms of last week.

Axiom 1 (Extensionality)**.** *Given sets x and y, we say $x = y \iff \forall a (a \in x \iff a \in y)$.*

Axiom 2 (Pairing)**.** *Given sets x and y, one can construct the set that has only x and y as elements. Such a set is written as $\{x, y\}$ or $\{y, x\}$.*

Axiom 3 (Union)**.** *Given set x, one can construct the set y such that $\forall a (a \in y \iff \exists r \in x (a \in r))$. The set y is called $\bigcup x$, the "union of x".*

Here are some more.

Axiom 4 (Existence)**.** *There exists a set. This is a pretty basic and easy axiom; typically, we say that the empty set exists, just to make it easier, so we can have a clear beginning.*

Axiom 5 (Separation)**.** *Given any set and any logical property about sets, one can construct a subset of the original set containing only those elements that satisfy the property. Such a set is written in* set-builder notation*, like this: $\{t \in x | p(t)\}$ (x is the original set, and $p(t)$ is the logical property). Here's one example: $\{t \in \mathbb{N} | t < 100\}$, read as "the set of elements t of $\mathbb{N}$ such that t is less than 100" or "t in $\mathbb{N}$ such that t is less than 100". You can think of it as an axiom that lets you build sets using other sets plus logic, via a formula that goes like $\{set | logic\ constraint\}$.*

Let's put these axioms to use. Start with the empty set $\emptyset$, and use the axiom of pairing to construct the set that contains the empty set and the empty set, i.e. the set $\{\emptyset, \emptyset\}$. By extensionality axiom, this set is equal to $\{\emptyset\}$. Now use the axiom of pairing on $\emptyset$ and $\{\emptyset\}$ to construct the set $\{\emptyset, \{\emptyset\}\}$. We now have three different sets that we can work with to build even more sets.

3. Ordinal Numbers

Define the number 0 to be the empty set - you can think of it as a convenient name for the empty set. Let 1 refer to the set $\{\emptyset\}$. Let 2 refer to the set $\{\emptyset, \{\emptyset\}\}$, which we can also write as $\{0, 1\}$. Let 3 refer to the set $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, which we can write as $\{0, 1, 2\}$.

We can keep going: in general, define the set n as $n = \{0, 1, 2, ..., n-1\}$.

Why is this useful? Notice a few things:

- Each set n has exactly n elements in it. This makes it intuitive why the set n deserves to be called n. For example, 5 has five elements in it.
- For any two numbers m and n, if $m < n$, then $m \in n$. For example, 4 is an element of 7.
- For any element $m \in n$, m is a subset of n. For example, 3 is an element of 5, and $3 = \{0, 1, 2\}$ is a subset of $5 = \{0, 1, 2, 3, 4\}$. Every other element of 5 is also a subset of 5. (In fact, they are all proper subsets.)
- (Kicker number 1) The set element relation, $\in$, is a partial order relation on n.
- (Kicker number 2) n with order relation $\in$ is a well-ordered poset.

The numbers, defined in this way, are called ordinal numbers. Specifically, these are the ordinal natural numbers.

Definition 2 (Ordinal Number)**.** An ordinal number is a set where the element relation $\in$ turns it into a well-ordered poset. Ordinal numbers are also called ordinals.

Each of the natural numbers, defined above as $n = \{0, 1, ..., n-1\}$, is an ordinals. The set $\omega = \{0, 1, 2, ...\}$ of all natural numbers is also an ordinal. The set $\omega \cup \{\omega\}$ is an ordinal, it is called $\omega + 1$. The set $\{0, 1, 2, ..., \omega, \omega+1, \omega+2, ...\}$ is also an ordinal, it is called $\omega + \omega$, or 2ω.

4. Axiom of Choice

The last big concept for today is something known as the Axiom of Choice. I will state it, explain it, then demonstrate it by using it to prove a theorem.

Axiom 6 (Axiom of Choice)**.** *Given an infinite set x of nonempty sets, it is possible to produce a set that contains exactly one element from each set in x.*

Given infinitely many boxes of toys, it is possible to pick out exactly one toy from each box (so long as no box is empty), and drop the toys you picked into a separate, big box. The idea here is that you are able to 'choose' one object from each of the boxes of objects. This axiom is intuitive - if you have a collection of nonempty boxes, it should be possible to pick something from each. But the idea of doing this infinitely many times is what makes it difficult. How do you know that you won't fail along the way? What if every time you try to select toys, you'll only end up with a finite collection? The Axiom of Choice is a logical guarantee that there are no plot twists when we move to infinity - at least, when picking objects.

The Axiom of Choice is independent of the other axioms. Later in this course, I will explain what that means.

Let's use the Axiom of Choice now to do something useful. (In fact, it is one of the most widely studied axioms, you can find books on it.)

Theorem 3 (Well-Ordering Theorem)**.** *Any set can be endowed with a relation that turns it into a well-ordered set.*

Proof. The idea is that we start with a completely plain set with no notion of how the objects compare to each other, and we have to define the relation step by step, and make sure that it's a well-order relation. We'll do this as follows: let's start with a set x, and make a poset P by choosing the elements of x one by one and putting them into P.

To start, pick any element of x, and let that be the smallest element in P.

Next, pick any other element, and let that be the second smallest element in P.

Then, pick another, then another, etc., each time putting the element into P and declaring it to be larger than every element already in P (and smaller than every element that will come after it).

After we do this infinitely many times, though, we might still not have everything. For example, if we started with $\mathbb{N}$ as our set, and we started with the element 4 and each time we picked an element bigger than 4, then we could go on forever but we'd still be missing the elements 0, 1, 2, and 3. This is where transfinite induction comes in: after infinitely many steps, we can keep going, and do more induction. So we make the inductive step as follows:

Inductive step: As long as $x \setminus P$ is nonempty, choose an element of x, add it to P, and declare it to be bigger than everything else currently in P.

We end the process when we are no longer able to continue. This process guarantees that, by the time we stop, P contains the entire set x, because we can only stop once we have everything.

In order to pull this off, we have to make infinitely many arbitrary choices. We must use the Axiom of Choice to guarantee that this is possible, and that we are therefore able to continue until we have to stop.

We now have a poset P whose set is x. Why is it well-ordered? Well, given any subset $S \subset P$, there is an element that was added before all the other elements were added; this element will be the minimum of S. Think about it this way: initially, nothing in S was picked up by the inductive process (because we started with nothing). Then, at some point, we started having stuff in S. The only way to go from having nothing in S to having something in S is to pick up stuff in S - but, we only ever add elements one at a time, therefore the transition from $P \cap S = \emptyset$ to $P \cap S \neq \emptyset$ had to happen at exactly one element. This element is the minimum of S. (It is possible to be more technically rigorous for this step of the proof, but it is quite a bit of work, and the intuition is the important part here.)

We have constructed the desired poset with set x. Thus x has been well-ordered, using the relation given by P. $\square$

5. Cartesian Product

Let's wind down with something (hopefully) simpler. An ordered pair is a collection of two objects in which the given order matters. Usually they're written with parentheses, like this: (x, y) or (a, b). We say the element on the left is the first element, and the one on the right is the second, or the last, element.

Ordered pairs are useful for a lot of things. If you want to pick stuff from two sets, for example, but it matters which object comes from which set, you'll need ordered pairs. If you want to describe xy-coordinates, then it matters which coordinate is which, so you need ordered pairs. If you want to list some values of a function, then it matters which element is in the domain and which is in the range, so you need ordered pairs. etc.

Given sets a and b, the Cartesian Product of a and b, written $a \times b$, is the set of all ordered pairs where the first element is from a and the second element is from b.

Example: the Cartesian product of $2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$ is the set $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$.

And that's it for the week!

6. Poset Examples

Batch 4.

- $\mathbb{Z}^-$ (pronounced "Z minus") is the set of negative integers under the usual order relation.
- $\mathbb{Z}V$ is the set $\mathbb{Z}$ with the following order relation: given two integers x, y, we have $x < y$ if $x/y < 1$ and $xy > 0$. (Here x/y means x divided by y.)
- $\mathbb{X}$ is the set $\mathbb{Z}^2$ of ordered pairs of integers (x_1, x_2) with the following order relation: $(x_1, x_2) < (y_1, y_2)$ if $(x_1 < y_1) \wedge (x_2 < y_2)$.
- $\mathbb{Y}$ is the set $\mathbb{Z}^2$ of ordered pairs of integers (x_1, x_2) with the following order relation: $(x_1, x_2) < (y_1, y_2)$ if $(x_1 < y_1) \vee (x_1 = y_1 \wedge x_2 < y_2)$.
- $\mathbb{M}$ is the set $\mathbb{Q} \times \mathbb{Z}$ of pairs of numbers (x_1, x_2) (where x_1 is a rational number and x_2 is an integer) with the following order relation: $(x_1, x_2) < (y_1, y_2)$ if $(x_1 < y_1) \vee (x_1 = y_1 \wedge x_2 < y_2)$.

Batch 5. Given a poset P with order relation $<_p$ and poset Q with order relation $<_q$, the *lexicographic ordering* on the cartesian product $P \times Q$ is the order relation defined as follows: given (a, x) and (b, y) in $P \times Q$, we have $(a, x) < (b, y)$ if $(a <_p b) \vee (a = b \wedge x <_q y)$. It's like a priority ordering, where the first coordinate is the most important one, and the second coordinate is only used as a tiebreaker.

- $n\omega$ is the lexicographic ordering on $n \times \omega$.
- $\omega \cdot n$ is the lexicographic ordering on $\omega \times n$.
- ω^2 is the lexicographic ordering on $\omega \times \omega$.