BEYOND INFINITY 1: POSETS

JADE NINE

Review: Sets

We will go into much more detail about set theory in the next weeks of this course, as sets are central to the study of logic. However there are some concepts you must know already, here is a brief review:

- A set is a collection of objects, which are called the elements of the set; if A is a set and x is an element of A, we write $x \in A$.
- The set with no elements in it is called the empty set, written $\emptyset$.
- If A, B are sets such that $\forall x((x \in B) \implies (x \in A))$, then B is said to be a subset of A, written $B \subset A$. (Note that the empty set is always a subset of every other set.)
- If A, B are sets such that $B \subset A$, and there are elements of A that are not in B, then B is said to be a proper subset of A. In this case A has everything that B has and more.
- Given sets A and B, the set $A \setminus B$, called A minus B, is the set of everything in A that is not in B. Note that B need not be a subset of A, any elements that B has that A does not have are simply ignored.
- Given elements of a set, one can write the set using curly brackets, like this: $\{0, 1, 2, 3\}$ is the set with the numbers 0, 1, 2, and 3 in it.

You should also know the following specific sets.

- $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of natural numbers.
- $\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3, ...\}$, the set of integers.
- $\mathbb{Q}$, the set of rational numbers, expressed as fractions of integers with nonzero denominator.

1. Partially Ordered Sets

A partially ordered set, also called a poset, is a set P together with a relationship between objects, usually written as $<$, that satisfies a few properties:

- Binary Relation: The relation $<$ relates two objects, and is a logical relation: for any $x, y \in P$, either $x < y$ is true or $\neg(x < y)$ (written $x \not< y$) is true.
- Irreflexive: $\forall x \in P(x \not< x)$
- Antisymmetric: $\forall x, y \in P((x < y) \implies (y \not< x))$
- Transitive: $\forall x, y, z \in P(((x < y) \wedge (y < z)) \implies (x < z))$

It is important to realize that this is an abstract definition; some of the common notions or intuitions you may have about the less than symbol might not necessarily be true in an arbitrary poset. For example, given two elements x, y in a poset P, they might not be related at all - it is possible that $x \not< y$ *and* $y \not< x$. In this case x and y are said to be incomparable.

Also, the ordering relation could hypothetically be anything; it is possible to put different relations on the same set, for example, and the result is two entirely different posets. For example, consider the simple set $\{0, 1\}$ which has two elements (the numbers 0 and 1), there are several different possible orders you could put on this set:

- Set $\{0, 1\}$ with order $0 < 1$
- Set $\{0, 1\}$ with order $1 < 0$
- Set $\{0, 1\}$ with empty order relation - this means that 0 and 1 are incomparable

The ordering relation adds some sort of shape or structure to the set, and one can come up with interesting properties that distinguish different types of partially ordered sets. Let's explore!

I'll present some properties of posets that we can use to tell different "shapes" apart from each other. Then, I'll give a few concrete examples of posets, with names, and let you explore their properties and compare them to each other.

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2. Poset Properties

First, an element x of a poset P is...

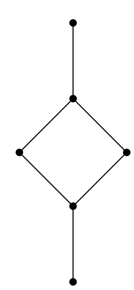
- An upper bound of P if for all elements $y \in P$, either $y < x$ or $y = x$.
- A lower bound of P if for all elements $y \in P$, either $x < y$ or $y = x$.
- Maximal if there is no element $y \in P$ such that $x < y$.
- Minimal if there is no element $y \in P$ such that $y < x$.

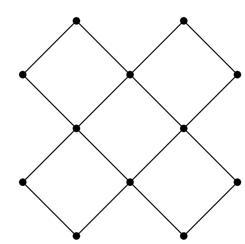
Now, a poset P is...

- Finite, if its set is finite.
- Infinite, if its set is infinite.
- Trivial, if all of its elements are incomparable to each other.
- Totally ordered, if all of its elements are comparable to each other - in other words, $\forall x, y \in P((x < y) \vee (y < x))$. (This is also sometimes called *linearly ordered*.)
- Connected if for any proper subset C of P, one can find an element x in $P \setminus C$ such that x is comparable with some element $y \in C$.
- Dense if for any two elements $x, y \in P$ where $x < y$, there is some element $z \in P$ such that $x < z < y$.
- Bounded if P has both an upper bound element and a lower bound element.

Recall that $x, y \in P$ are comparable if $x < y \vee y < x$.
We say that two elements $x, y \in P$ are compatible if there exists some element z such that $z < x \wedge z < y$.
Note that x, y do not have to be comparable to be compatible.

Dot-and-line diagrams. *(Also called Hasse diagrams.)* Posets can be drawn in diagrams, using dots and lines, where the dots represent elements of the set, and lines connect dots that are comparable, in such a way that dots lower down are less than dots higher up. The exact height difference does not matter, as long as you place the lesser element lower than the greater element. Additionally, you don't have to draw lines between every pair of comparable elements, as long as there's an indirect connection going entirely up from one element to another element, then you know they are comparable. Note that if there is no way to get from one element to another without zigzagging up and down or hopping between dots not connected by lines, then the elements are incomparable. Here are some examples of dot and line diagrams:





Properties about posets can be clearer or more apparent from a diagram than from the poset's definition.
Also, drawing diagrams helps you judge your own understanding of a poset, as you can judge the quality of your own diagram.

Lastly, if you can draw diagrams for two posets in such a way that the diagrams end up looking the same, then those posets are said to have the same shape. We will define this more precisely next week; for now, it is enough to understand that two posets defined in different ways might end up producing the same diagram.

3. Poset Examples

In order to define a poset, I have to tell you both the set and the order relation. That said, the names for all the following will refer to the whole poset, both set and order relation.

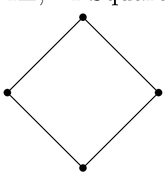
These names are not standard names, rather I made names for convenience so that the posets are easier to talk about and remember, and so that you can use these named objects to structure your thinking. Some of these posets do have standard names, which we will cover later; some of them don't.

Batch 1.

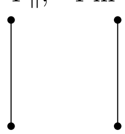
- $2\uparrow$ is the set $\{0, 1\}$ with $0 < 1$
- $2\downarrow$ is the set $\{0, 1\}$ with $1 < 0$
- $2\cdot$ is the set $\{0, 1\}$ with 0 and 1 incomparable
- $\mathbb{N}+$ is the set $\mathbb{N}$ with the usual order relation, where $x < y$ if $x - y$ is a negative integer.
- $\mathbb{N}\times$ is the set $\mathbb{N}$ with the divisibility order relation, where $x < y$ if x is a *factor* of y. *(Note: we consider every number to be a factor of* 0*, and* 0 *to be a factor of nothing but* 0 *itself. Also,* 1 *is a factor of every number.)*
- $\mathbb{Z}+$ is the set $\mathbb{Z}$ with the usual order relation, where $x < y$ if $x - y$ is a negative integer.
- $\mathbb{Q}+$ is the set $\mathbb{Q}$ with the usual order relation, where $x < y$ if $x - y$ is a negative rational number (a negative rational number is a ratio of two integers p, q where exactly one of p, q is negative).
- $\overline{\mathbb{Q}}+$, "Q-bar plus", is the set of all rational numbers between 0 and 1 (inclusive) with the usual order relation.

Batch 2, here are some posets that each have 4 elements but have different order relations. They're given as dot-and-line diagrams.

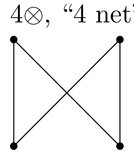
$4\square$, "4 Square".



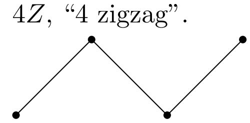
$4\parallel$, "4 in two lines".



$4\otimes$, "4 net".



$4Z$, "4 zigzag".



Batch 3.

- $\mathbb{S}2$ is the set of all subsets of $\{0, 1\}$ where a subset A is less than a subset B if A is a proper subset of B.
- $\mathbb{S}3$ is the set of all subsets of $\{0, 1, 2\}$ where a subset A is less than a subset B if A is a proper subset of B.
- *Cube* is a poset on 8 elements: take a cube and position it so that it is standing on one of its vertices. Then the 8 vertices are the elements of the poset, and any time two vertices are connected by an edge, the vertex at the edge's lower endpoint is less than the vertex at the edge's higher endpoint.
- $\aleph$, pronounced "aleph", is the set of all strings of letters ordered alphabetically. The strings of letters do not have to be real words; it is still possible to order nonsense alphabetically. The strings do each have to contain a finite number of letters, though.
- $\beth$, pronounced "bet", is the set of all strings of letters ordered by substring: one string is less than another if it is a substring of the other, or "fits inside" the other. For example, "pear"$<$"pearl", "par"$<$"apparent", "sin"$<$"praising", "meow"$<$"homeowner", "bub"$<$"bubble", etc. In any other case, the strings are incomparable: "ant" is incomparable with "xyz", "sunrise" is incomparable with "sunlight", "pat" is incomparable with "part", etc.

Batch 4. You can ignore these for now, this batch is just a taster, we will discuss these more next week.

- $\mathbb{Z}V$ is the set $\mathbb{Z}$ with the following order relation: given two integers x, y, we have $x < y$ if $x/y < 1$ and $xy > 0$. (Here x/y means x divided by y.)
- $\mathbb{X}$ is the set $\mathbb{Z}^2$ of ordered pairs of integers (x_1, x_2) with the following order relation: $(x_1, x_2) < (y_1, y_2)$ if $(x_1 < y_1) \wedge (x_2 < y_2)$.
- $\mathbb{Y}$ is the set $\mathbb{Z}^2$ of ordered pairs of integers (x_1, x_2) with the following order relation: $(x_1, x_2) < (y_1, y_2)$ if $(x_1 < y_1) \vee (x_1 = y_1 \wedge x_2 < y_2)$.

4. Problems

1. (a) For each of the posets in batch 1, draw a dot-and-line diagram (you don't have to draw infinitely many dots for the infinite ones, just enough of the diagram to get a sense of what it looks like). You can label the dots if it helps you, but this is not necessary.
(b) Look at your diagrams for $2\uparrow$ and $2\downarrow$. Do they look the same? If not, is it possible to draw them so that they look the same?
(c) Draw a dot-and-line diagram for $\mathbb{S}2$. Don't forget to include the empty set and the set $\{0,1\}$ itself (these both count as subsets of $\{0,1\}$)! Use your diagram to help you find the poset from batch 2 that has the same shape as $\mathbb{S}2$.
(d) Can you find any more posets among the first three batches that have the same shape? *[Hint: There is one more pair to find.]*

2. Please explain or prove your answers for all questions in this problem.
(a) One clear way to tell posets apart is to test for properties and then use that to distinguish different posets. Prove that $4\square$ is bounded and $4\parallel$ is not.
(b) Among the four posets of batch 2, determine which of them are connected.
(c) Of the properties described in the poset properties section, which one could you use to distinguish $\mathbb{N}+$ from $\mathbb{N}\times$? *(There is more than one that works, you just have to find one.)*
(d) Of the properties described in the poset properties section, which one could you use to distinguish $\aleph$ from $\beth$?
(e) Describe all minimal elements in $\aleph$. Then, describe all minimal elements in $\beth$.
(f) "Puncturing" is the technique of removing one element of a mathematical object and studying the properties of the result. Prove that $4Z$ can be punctured to produce a disconnected poset, but $4\otimes$ cannot. (This proves that $4Z$ and $4\otimes$ must therefore be different posets.)

3. A fun logical puzzle is noticing that there is a relationship between the irreflexive, antisymmetric, and transitive properties of posets.
(a) Prove that, given a set P with a binary relation that is irreflexive and transitive, that the relation must also be antisymmetric.
(b) Prove that, given a set P with a binary relation that is antisymmetric, that the relation must also be irreflexive. (You don't need transitivity to prove this.)

4. Posets will be useful for us because of just how abstract and arbitrary they can be. For example, it is possible to add "artificial" elements to a poset, which essentially means inventing a new element, putting that element in the set, and then declaring what its relationship to all the other elements shall be.

Suppose we take $\mathbb{Q}+$ and add two new elements, one called l for "very low" and the other called h for "very high", where for all rational numbers $x \in Q$, we declare that $l < x$ and $x < h$. We can call the result $\mathbb{Q}?$, or "Q question-mark".
(a) Prove that l is a lower bound of $\mathbb{Q}?$ and h is an upper bound of $\mathbb{Q}?$.
(b) Go through each of the properties in the poset properties section (just the ones that apply to posets) and figure out whether they're true for $\mathbb{Q}?$.
(c) Go through each of the properties in the poset properties section (just the ones that apply to posets) and figure out whether they're true for $\overline{\mathbb{Q}}+$.

5. (a) A collection of elements from a poset is said to be mutually incomparable if every element in the collection is incomparable with every other element. For example, the two minimal elements in $4\parallel$ form a mutually incomparable collection. What's the largest collection of mutually incomparable elements you can find in $Cube$?
(b) A collection of elements from a poset is defined as mutually incompatible in a similar way. What's the largest collection of mutually incompatible elements you can find in $\beth$?
(c) What's the largest collection of mutually *incomparable* elements you can find in $\beth$?

6. This problem will have you create some posets of your own.
(a) Draw a few dot-and-line diagrams for various posets on 5 elements. Give them names of your own.
(b) Draw a few dot-and-line diagrams for various posets on 5 elements that are disconnected (you can refer back to the definition of connected in order to do this). Give them names of your own.

(c) Draw a dot-and-line diagram for a poset on 5 elements that is bounded but not totally ordered.
(d) Come up with an infinite poset that is bounded but not totally ordered, and describe how to define your poset.
(e) Come up with a poset that is dense but not totally ordered, and describe how to define your poset.
(f) Come up with a poset that is connected but becomes disconnected if you puncture it anywhere.