March 23, 2025 Math 9

Algebra.

Powers and roots.

Integer powers. For any integer $n, m \in \mathbb{Z}$,

$$a^n \cdot a^m = a^{n+m}, \qquad \frac{a^n}{a^m} = a^n \cdot a^{-m} = a^{n-m},$$

$$(a^n)^m = a^{n \cdot m} = (a^m)^n \ (\forall\, n, m \in \mathbb{Z}).$$

Algebraic roots. For any integer $m \in \mathbb{Z}$ and natural $n \in \mathbb{N}$, $a, b \in \mathbb{R}_+, c \in \mathbb{R}$:

- $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$
- $\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}$ $(m > 0)$
- $\sqrt[n]{a} = \sqrt[n \cdot m]{a^m}$ $(m > 0)$
- $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ $(a \neq 0\ if\ m \leq 0)$
- $\sqrt[m]{(-a)^m} = a\ if\ m = 2k, \sqrt[m]{(-a)^m} = -a, if\ m = 2k + 1$

Rational powers. For any integer $p \in \mathbb{Z}$ and natural $q \in \mathbb{N}$,

$$a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{a}\right)^p \ (a \in \mathbb{R}_+, q \in \mathbb{N}, p \in \mathbb{Z}),$$

defines power for rational values of exponent. The following rules apply in this case, which follow from the above properties of integer powers and roots.

- $(ab)^p = a^p b^p$
- $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$
- $a^p \cdot a^q = a^{p+q}$
- $(a^p)^q = a^{pq}$
- $(a^p)^{\frac{1}{q}} = a^{\frac{p}{q}}$

Intervals of monotonic behavior. For $a > 1$ the value of a^p increases when p increases. For $0 < a < 1$ the value of a^p decreases when p increases. For rational $p = m/n$ this can be straightforwardly proven by finding the common denominator of $p = m/n < q = r/s$ (case of positive and negative p should be considered separately).

Consequently, we can extend the definition of powers to irrational numbers x, such as $\sqrt{2}$, as follows.

Definition. For an irrational $x \in R$, and $a > 1$, a^x is a number such that that for any rational q less than x, $a^x > a^p$, while for any rational number greater that x, $a^x < a^p$,

$$a^x > a^p, \forall p < x, p \in \mathbb{Q}, a > 1$$
$$a^x < a^p, \forall p > x, p \in \mathbb{Q}, a > 1$$

Similarly, for $0 < a < 1$,

$$a^x < a^p, \forall p < x, p \in \mathbb{Q}, 0 < a < 1$$
$$a^x > a^p, \forall p > x, p \in \mathbb{Q}, 0 < a < 1$$

It is important to mention that in order to make this definition consistent we must prove that such a number exists and is unique (eg via Dedekind section).

Now, using the above definition we have a way to calculate, say, $2^{\sqrt{2}}$, to any given accuracy. In order to do so, we must simply find a rational number p that is close enough to $\sqrt{2}$ and compute a^p. In order to improve the accuracy, we may choose another number, q, yet closer to $\sqrt{2}$, and use it for the computation, and so on. We can obtain a sequence of rational numbers approaching $\sqrt{2}$ (and $\sqrt{p}$ for any rational p) by using the continuous fraction,

$$\sqrt{2} = a + \cfrac{c}{b + \cfrac{c}{b + \cfrac{c}{b + \cdots}}}$$

Exercise. What are the coefficients a, b, and c here?

Solution of some homework problems.

1. Compare the following real numbers (are they equal? which is larger?)
   a. 1.33333... = 1.(3) and 4/3

$$1.33333\ldots = 1 + \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right) = 1 + \frac{3}{10}\frac{1}{1-\frac{1}{10}} = 1 + \frac{1}{3} = \frac{4}{3}.$$

   b. 0.09999... = 0.0(9) and 1/10

$$0.09999\ldots = 9\left(\frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots\right) = \frac{9}{100}\frac{1}{1-\frac{1}{10}} = \frac{1}{10} = 0.1$$

   c. 99.9999... = 99.(9) and 100

$$99.9999\ldots = 90 + 9\left(1 + \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right) = 90 + 9\frac{1}{1-\frac{1}{10}} = 100.$$

   d. $\left(\sqrt[2]{2} < \sqrt[3]{3}\right) \Leftrightarrow (2^3 < 3^2) \Leftrightarrow (8 < 9)$
2. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/8

$$\frac{1}{8} = \frac{1}{2^3} = 0.001B.$$

   b. 1/7

$$\frac{1}{7} = \frac{1}{8}\frac{1}{1-\frac{1}{8}} = \frac{1}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right) = 0.001001001\ldots B = 0.(001)B.$$

   c. 2/7

$$\frac{2}{7} = 2 \cdot \frac{1}{7} = 2 \cdot 0.001001001\ldots B = 0.01(001)B.$$

   d. 1/6

$$\frac{1}{6} = \frac{1}{8}\frac{1}{1-\frac{1}{4}} = \frac{1}{2^3}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right) = 0.0010101\ldots B = 0.001(01)B.$$

   e. 1/15

$\frac{1}{15} = \frac{1}{16}\frac{1}{1-\frac{1}{16}} = \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{2^8} + \frac{1}{2^{12}} + \cdots\right) = 0.000100010001 \ldots B = 0.(0001)B.$

f. 1/14

$\frac{1}{14} = \frac{1}{16}\frac{1}{1-\frac{1}{8}} = \frac{1}{2^4}\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right) = 0.0001001001 \ldots B = 0.0001(001)B.$

g. 0.1

$\frac{1}{10} = \frac{1}{8}\frac{1}{1+\frac{1}{4}} = \frac{1}{2^3}\left(1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \cdots + \frac{1}{2^{2n}} - \frac{1}{2^{2n+2}} + \cdots\right) = \frac{1}{2^3}\left(\frac{3}{2^2} + \frac{3}{2^6} + \frac{3}{2^{10}} + \cdots + \frac{3}{2^{4n+2}} + \cdots\right) = \frac{1}{2^3}\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{10}} + \cdots + \frac{1}{2^{4n+1}} + \frac{1}{2^{4n+2}} + \cdots\right) = 0.0001100110011 \ldots B = 0.00011(0011)B,$
or, using the base multiplication,

$2 \times 0.1 = 0.2 \Rightarrow 0.1 = 0.0 \ldots B,$

$2 \times 0.2 = 0.4 \Rightarrow 0.1 = 0.00 \ldots B,$

$2 \times 0.4 = 0.8 \Rightarrow 0.1 = 0.000 \ldots B,$

$2 \times 0.8 = 1 + 0.6 \Rightarrow 0.1 = 0.0001 \ldots B,$

$2 \times 0.6 = 1 + .2 \Rightarrow 0.1 = 0.00011 \ldots B = 0.00011(0011)B.$

h. 0.33333... = 0.(3)

$0.33333 \ldots = \frac{1}{3} = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots\right) = 0.010101 \ldots B = 0.(01)B.$

i. 0.13333... = 0.1(3)

$0.133333 \ldots = \frac{4}{30} = \frac{2}{15} = \frac{1}{4}\frac{1}{1-\frac{1}{16}} = \frac{1}{2^2}\left(1 + \frac{1}{2^4} + \frac{1}{2^8} + \frac{1}{2^{12}} + \cdots\right) = 0.0100010001 \ldots B = 0.01(0001)B.$