Homework for January 5, 2025

Geometry.

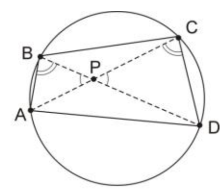
Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proofs of Ptolemy's and Euclid's theorems. Try solving the following problems including the unsolved problems from previous homeworks (problems marked with asterisk are optional – you are not expected to solve them).

Problems.

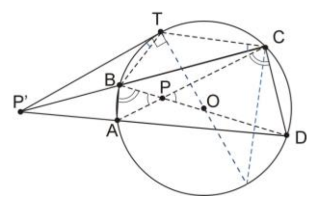
1. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

   $$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$$

   where $|PT|$ is a segment tangent to the circle (see Figure).

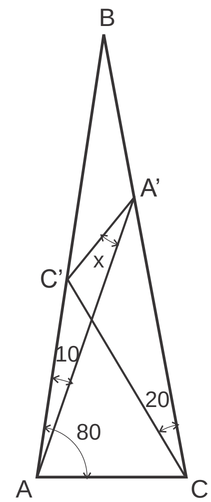


2. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.



   a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
   b. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?
3. Using the Ptolemy's theorem, prove the following:
   a. Given an equilateral triangle $\triangle ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
   c. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle?

4. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.
5. Consider all triangles with a given base and given altitude corresponding to this base. Prove that among all these triangles the isosceles triangle has the biggest angle opposite to the base.
6. Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. *In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 20°$ and $\angle BAA' = 10°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



8. * Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).

Algebra.

Review the previous classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems (some problems are repeated from the previous homework – skip the ones you have already solved).

1. Rewrite the following properties of set algebra and partial ordering operations on sets in the form of logical propositions, following the first example.
   a. $[\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) = \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}] \Leftrightarrow \big[(\mathrm{x} \in \mathrm{A}) \wedge \big((\mathrm{x} \in \mathrm{B}) \vee (\mathrm{x} \in \mathrm{C})\big)\big] = \big[\big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{B})\big) \vee \big((\mathrm{x} \in \mathrm{A}) \wedge (\mathrm{x} \in \mathrm{C})\big)\big]$
   b. $A + (B \cdot C) = (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$

g. $(A \subset B) \Leftrightarrow (B' \subset A')$
h. $(A + B)' = A' \cdot B'$
i. $(A' + B')' + (A' + B\,)' = A$

2. Using definitions from the classwork handout, devise logical arguments proving each of the following properties of algebra and partial ordering operations on sets and draw Venn diagrams where possible (hint: use problem #1).
   a. $\mathrm{A} \cdot (\mathrm{B} + \mathrm{C}) \;=\; \mathrm{A} \cdot \mathrm{B} + \mathrm{A} \cdot \mathrm{C}$
   b. $A + (B \cdot C) \;=\; (A + B) \cdot (A + C)$
   c. $(A \subset B) \Leftrightarrow A + B = B$
   d. $(A \subset B) \Leftrightarrow A \cdot B = A$
   e. $(A + B)' = A' \cdot B'$
   f. $(A \cdot B)' = A' + B'$
   g. $(A \subset B) \Leftrightarrow (B' \subset A')$
   h. $(A + B)' = A' \cdot B'$
   i. $(A' + B')' + (A' + B\,)' = A$
3. Verify that a set of eight numbers, {1,2,3,5,6,10,15,30}, where addition is identified with obtaining the least common multiple,

$$m + n \equiv LCM(n, m)$$

multiplication with the greatest common divisor,

$$m \cdot n \equiv GCD(n, m)$$

$m \subset n$ to mean "m is a factor of n",

$$m \subset n \equiv \big(n = 0 mod(m)\big)$$

and

$$n' \equiv 30/n$$

satisfies all laws of the set algebra.

4. For a set A, define the characteristic function χ_A as follows,

$$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x\ \notin A \end{cases}$$

Show that χ_A has following properties

$$\chi_A = 1 - \chi_{A'}$$

$$\chi_{A\cap B} = \chi_A \chi_B$$

$$\chi_{A\cup B} = 1 - \chi_{A'\cap B'} = 1 - \chi_{A'}\chi_{B'} = 1 - (1 - \chi_A)(1 - \chi_B)$$
$$= \chi_A + \chi_B - \chi_A\chi_B$$

Write formulas for $\chi_{A\cup B\cup C}$, $\chi_{A\cup B\cup C\cup D}$.

5. Consider the quadratic equation $x^2 = 7x + 1$. Find a continued fraction corresponding to a root of this equation.

6. Using the continued fraction representation, find rational number, r, approximating $\sqrt{2}$ to the absolute accuracy of 0.0001.

7. Consider the values of the following expression, y, for different x. How does it depend on x when n becomes larger and larger?

$$n\ fractions \left\{ \quad y = 3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cdots}}} \cdots - \frac{2}{3-x} \right. .$$