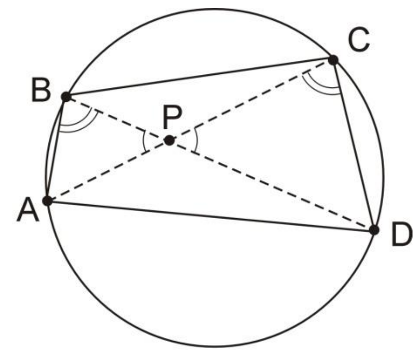
December 8, 2024 Math 9

# Geometry.

### Corollaries of the Inscribed Angle Theorem. Euclids' theorems. Power of a point to a circle.

### Euclids' theorems (Proposition 35 of Book 3 of Euclid's Elements).

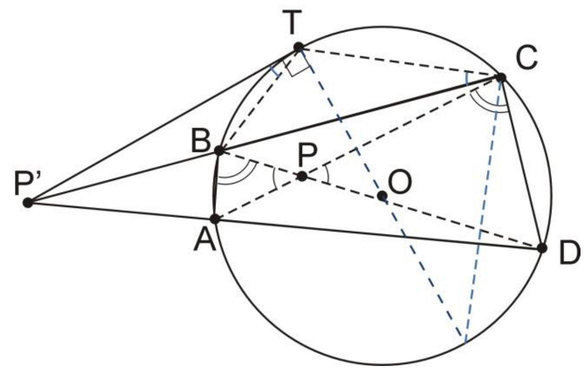
Consider the following figures. Using the theorem on the angle inscribed into a circle and the similarity of the corresponding triangles, it is easy to prove the following Euclid theorems.




i. If two chords, $AC$ and $BD$ intersect at a point $P$ inside the circle, then,

$|AP||PC| = |BP||PD| = R^2 - d^2,$

where $R$ is the radius of the circle and and d is the distance from point $P$ to the center of the circle, $d = |PO|$.




**Proof**. $\Delta APB \sim \Delta DPC$, so $\frac{|AP|}{|BP|} = \frac{|PD|}{|PC|}$, or, $|AP||PC| = |BP||PD| = R^2 - d^2$.

ii. If two chords, $AD$ and $BC$ intersect at a point $P'$ outside the circle, then,

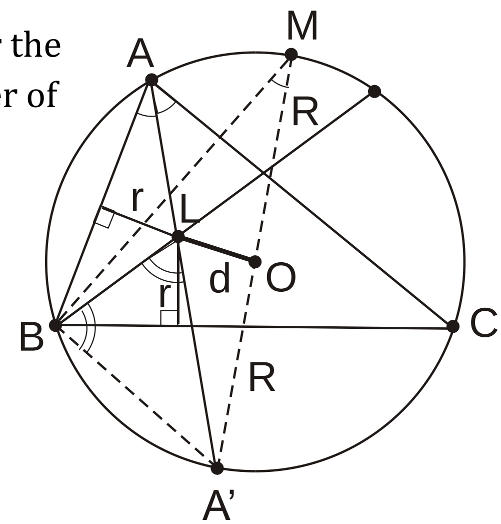
$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$

where $|PT|$ is a segment tangent to the circle.

**Proof**. $\Delta P'BD \sim \Delta P'AC$, so $\frac{|P'A|}{|P'B|} = \frac{|P'D|}{|P'C|}$, or, $|P'A||P'D| = |P'B||P'C|$.

For any circle of radius $R$ and any point $P$ distant $d$ from the center, the quantity $d^2 - R^2$ is called the power of $P$ with respect to the circle.

## Application of the Euclid’s theorems: Euler’s formula.




Using the above theorem, the following formula for the distance between the incenter and the circumcenter of a triangle can be established.

Let $O$ and $L$ be the circumcenter and the incenter (that is, center of the circumscribed and the inscribed circle), respectively, of a triangle $ABC$, with circumradius $R$ and inradius $r$. Then, the distance $|OL| = d$ is given by,

$d^2 = R^2 - 2Rr.$

Indeed, consider the figure, where the chord $AA'$ passes through the incenter $L$, and the chord $A'M$ is the diameter of the circumcircle, passing through its center $O$. Triangle $A'MB$ is the right triangle by the inscribed angle theorem, and by the same theorem $\angle BAA' = \angle BMA'$. Hence, $\Delta A'BM$ is similar to the triangle with the hypotenuse $AL$ whose leg is the radius of the inscribed circle (cf. Figure), so

$|A'M|:|A'B| = |AL|:r.$

Note that triangle $BA'L$ is isosceles, and therefore $|A'B| = |A'L|$. This is because $\angle A'LB = \angle ABL + \angle BAL$ as an external angle of $\Delta ABL$, while $\angle A'BL = \angle A'BC + \angle CBL = \angle A'AC + \angle CBL$ by the inscribed angle theorem, and $\angle BAL = \angle A'AC$ and $\angle ABL = \angle CBL$ since $AL$ and $BL$ are bisectors of $\angle BAC$ and $\angle CBA$, respectively (because $L$ is the incenter).

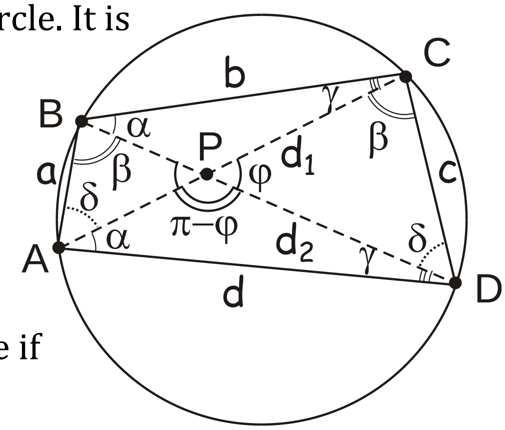
Substituting $|A'B| = |A'L|$ and $|A'M| = 2R$ in the above and using the Euclid theorem, $|AL||A'L| = R^2 - d^2$, we obtain,

$|AL||A'B| = |AL||A'L| = R^2 - d^2 = 2Rr,$

which proves the above Euler’s formula.

**Properties of inscribed quadrilaterals. Ptolemy's theorem.**

Consider the quadrilateral $ABCD$ inscribed into a circle. It is clear from the theorem on the inscribed angle that the opposite angles of $ABCD$ are supplementary (i. e. add to 180 degrees),




$$\hat{A} + \hat{C} = \hat{B} + \hat{D} = \pi$$

**Theorem.** A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

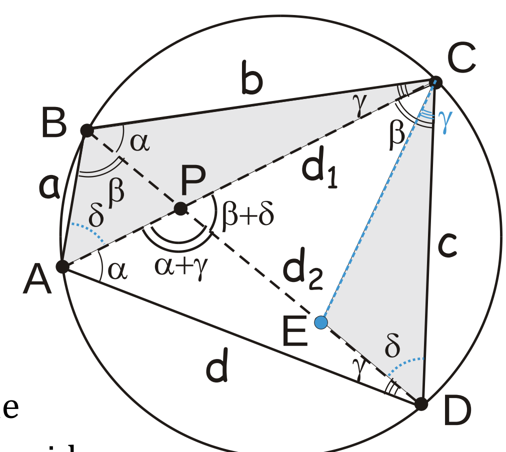
Now consider angles $\alpha, \beta, \gamma, \delta$, between the sides and the diagonals. The angle between the diagonals, $\varphi = \alpha + \gamma = \pi - (\beta + \delta)$.

**Theorem (Ptolemy).** A quadrilateral can be inscribed in a circle if and only if the product f its diagonals equals the sum of the products of its opposite sides,

$$d_1 d_2 \;=\; ac + bd \qquad (1)$$

<u>Proof of the necessary condition of Ptolemey's theorem</u>, i.e. of Eq. (1) for an inscribed quadrilateral.

Geometrical proof employs an elegant supplementary construct. Inventing such an additional geometrical element is one of the key, most important and powerful methods of geometrical proof.




Draw segment $CE$, whose endpoint, $E$, belongs to the diagonal BD, and which is at an angle $\gamma = \widehat{ACB}$ to the side $CD$. Thus obtained $\Delta DEC \sim \Delta ABC$. Therefore, $\frac{|AC|}{c} = \frac{a}{|ED|}$.

Furthermore, $\widehat{BCE} = \widehat{ACD} = \beta$ and therefore $\Delta BCE \sim \Delta ACD$, so $\frac{|AC|}{d} = \frac{b}{|BE|}$.
Adding thus obtained equalities, we get

$$ac + bd = |AC||ED| + |AC||BE| = d_1 d_2.$$

The sufficiency of this condition can be easily proven by contradiction.