Homework for October 27, 2024.

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework. Solve the following problems (you may skip the problems solved before or considered in class).

Recap. In order to prove the equality $A(n) = B(n)$, for any n, using the method of mathematical induction you have to

- Prove that $A(1) = B(1)$
- Prove that $A(k+1) - A(k) = B(k+1) - B(k)$ (*)
- Then from assumption $A(k) = B(k)$ and from equality (*) follows $A(k+1) = B(k+1)$

1. Using mathematical induction, prove that $\forall n \in \mathbb{N}$,
   a. $\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,
   b. $\sum_{k=1}^{n}(2k)^2 = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$
   c. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$
   d. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} < \frac{1}{2}$
   e. $\sum_{k=1}^{n} \frac{1}{(7k-6)(7k+1)} = \frac{1}{1\cdot 8} + \frac{1}{8\cdot 15} + \frac{1}{15\cdot 22} + \cdots + \frac{1}{(7n-6)(7n+1)} < \frac{1}{7}$
   f. $\sum_{k=n+1}^{3n+1} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1} > 1$
2. Prove by mathematical induction that for any natural number n,
   a. $5^n + 6^n - 1$ is divisible by 10
   b. $9^{n+1} - 8n - 9$ is divisible by 64
3. **Recap**. Binomial coefficients are defined by
   $C_n^k = {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   a. Prove that $C_{n+k}^2 + C_{n+k+1}^2$ is a full square
   b. Find n satisfying the following equation,
   $$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = 1023$$
   c. Prove that
   $$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = 2^{n-1}$$

4. Prove that binomial coefficients satisfy the following identities,

a. $C_n^0 = C_n^n \Leftrightarrow \binom{n}{0} = \binom{n}{n} = 1$

b. $C_n^k = C_n^{n-k} \Leftrightarrow \binom{n}{k} = \binom{n}{n-k}$

c. $C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

d. $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k \Leftrightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

e. $C_{n+1}^k = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

f. $C_n^{k+1} = \binom{n}{k+1} = \binom{n}{k}\frac{n-k}{k+1}$

g. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

5. Find all x satisfying the following equation:

$$\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0.$$

Geometry.

Review the classwork handout. Solve the remaining problems from the previous homework; consider solutions explained in the classwork handout. Try solving the following additional problems. In all the problems, you are only allowed to use theorems we had proven before.

Problems.

1. Prove that for any triangle ABC with sides a, b and c, the area, $S \leq \frac{1}{4}(b^2 + c^2)$.
2. In an isosceles triangle ABC with the side $|AB| = |BC| = b$, the segment $|A'C'| = m$ connects the intersection points of the bisectors, AA' and CC' of the angles at the base, AC, with the corresponding opposite sides, $A' \in BC$ and $C' \in AB$. Find the length of the base, $|AC|$ (express through given lengths, b and m).
3. Prove that for any point on a side of an equilateral triangle, the sum of the distances to the two other sides is the same constant. What is this distance (the side of the triangle is a)?

4. Distances from the point M inside an equilateral triangle ABC to the respective sides of this triangle are, d_a, d_b and d_c. Find the altitude of this triangle.
5. Three lines parallel to the respective sides of the triangle ABC intersect at a single point, which lies inside this triangle. These lines split the triangle ABC into 6 parts, three of which are triangles with areas S_1, S_2, and S_3. Show that the area of the triangle ABC, $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$.