

Homework for October 6, 2024.

## Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework. Solve the following problems (you may skip the ones considered in class).

1. Using logical equivalence,  $\sim(A \Rightarrow B) \Leftrightarrow (A \wedge \sim(B))$ , consider/prove,

$$\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$$

2. Let  $P, Q$  and  $R$  be some logical propositions. Write a negation of the logical expression and simplify the result as much as possible,

- $((P \wedge Q) \vee (R \wedge Q))$
- $((P \vee R) \wedge (Q \vee R))$
- $(P \Leftrightarrow (Q \wedge R))$
- $((P \wedge Q) \Rightarrow (Q \vee R))$
- $((P \vee Q) \Leftrightarrow (Q \vee R))$

3. Solve the following equations:

- $\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$
- $\sqrt{3x+4} + \sqrt{x-2} = 2\sqrt{x}$
- $\frac{1}{x^3+2} - \frac{1}{x^3+3} = \frac{1}{12}$
- $\frac{1}{x^2} + \frac{1}{(x+2)^2} = \frac{10}{9}$
- $\sqrt{x-2} = x-4$
- $1 + \sqrt{1 + x\sqrt{x^2 - 24}} = x$
- $\sqrt{x} + \frac{2x+1}{x+2} = 2$

4. Simplify expressions:

- $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$
- $\frac{\sqrt{\frac{x-1}{x+1} + \frac{x+1}{x-1}} - 2}{\sqrt{(x+1)^3} - \sqrt{(x-1)^3}} (2x + \sqrt{x^2 - 1})$
- $\frac{\sqrt{x-4\sqrt{x-4}+2}}{\sqrt{x+4\sqrt{x-4}-2}}$

5. Prove that:

a.  $\frac{\sqrt{7+4\sqrt{3}} \cdot \sqrt{19-8\sqrt{3}}}{4-\sqrt{3}} - \sqrt{3} = 2$

b.  $\sqrt{6a + 2\sqrt{9a^2 - b^2}} - \sqrt{6a - 2\sqrt{9a^2 - b^2}} = 2\sqrt{3a - b}$

6. **Recap.** There are 15 students in a class.

a. Each student needs to make a presentation of a problem. How many ways are there to arrange the order in which they do presentations – i. e. decide who speaks first, who speaks second, third, ..., fifteenth?

b. How many ways is there to select a pair of students of whom one will present solution of an algebra problem, and the other of a problem in geometry?

c. How many ways is there to select a team of two students who will represent the class at a math competition?

7. **Recap.** Binomial coefficients are defined by

$$C_n^k = {}_k C_n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Prove that binomial coefficients satisfy the following identities,

$$C_n^0 = C_n^n \Leftrightarrow \binom{n}{0} = \binom{n}{n} = 1$$

$$C_n^k = C_n^{n-k} \Leftrightarrow \binom{n}{k} = \binom{n}{n-k}$$

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k \Leftrightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$C_{n+1}^k = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$C_n^{k+1} = \binom{n}{k+1} = \binom{n}{k} \frac{n-k}{k+1}$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

## Geometry.

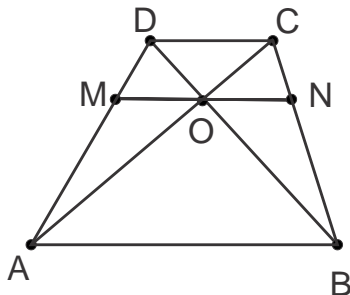
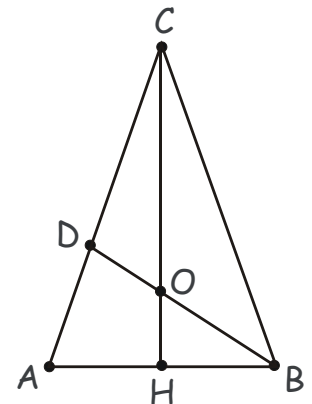
Review the classwork handout. Solve the remaining problems from the previous homework. In particular, review constructing the segments equal to the ratio and the product of two given segments. Try solving the following additional problems. Although we will review the Thales theorem next time – you may assume that it had been already proven.

### Problems.

1. Given two segments of integer lengths  $a$  and  $b$ , construct,
  - a. A segment of length  $a^2$
  - b. A segment of length  $1/a$
  - c. A segment of length  $ab$
  - d. A segment of length  $a/b$

Hint: construct these segments given  $a, b$ , and a segment of length 1.

2. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle “trisect” one another (Coxeter, Gretzer, p.8).
3. In isosceles triangle  $ABC$  point  $D$  divides the side  $AC$  into segments such that  $|AD|:|CD|=1:2$ . If  $CH$  is the altitude of the triangle and point  $O$  is the intersection of  $CH$  and  $BD$ , find the ratio  $|OH|$  to  $|CH|$ .
4. In a trapezoid  $ABCD$  with the bases  $|AB| = a$  and  $|CD| = b$ , segment  $MN$  parallel to the bases,  $MN \parallel AB$ ,



connects the opposing sides,  $M \in [AD]$  and  $N \in [BC]$ .  $MN$  also passes through the intersection point  $O$  of the diagonals,  $AC$  and  $BD$ , as shown in the Figure.

Prove that  $|MN| = \frac{2ab}{a+b}$ .