September 29, 2024

Algebra.

Elements of Mathematical Logic (continued). Predicate Calculus. Quantifiers.

Definition. A **predicate** with a variable is a **proposition**, if either (i), or (ii):

i. a value is assigned to the variable
ii. possible values of the variable are quantified using a **quantifier**

Example. For $x > 1$ to be a proposition, either we must substitute a specific number for x, or change it to something like "There is a number x for which $x > 1$ holds"; equivalently, using a quantifier, $\exists\, x, x > 1$.

Quantifiers.

$\exists$ is called the **existential quantifier**, and reads " … there exists …".

$\exists\, x \in X$: $\Leftrightarrow$ " … there exists an x in the set X such that …"

For example, "someone lives on a remote island" could be transformed into the propositional form, $\exists\, x: P(x)$, where:

- $P(x)$ is the predicate, stating: **x hibernates during the winter time**,
- Set of objects of interest X includes (not limited to) all living creatures.

The statement $D(x)$: "equation $x^3 + 3x^2 + 5x + 15 = 0$ has a real solution", can be written in a predicate form as: $\exists x \in R: x^3 + 3x^2 + 5x + 15 = 0$.

Exercise. Try to construct negation for $P(x)$ and $D(x)$.

$\forall$ is called the **universal quantifier**, and reads " … for all …".

$\forall x \in X$: $\Leftrightarrow$ " … for all x in the set X …"

Example 1. "All airplanes have wings" could be transformed into the propositional form, $\forall\{x, (x\ is\ an\ airplane)\}: D(x)$, where,

- $D(x)$ is the predicate stating: **x has wings**, and
- Set of objects of interest, X, is only populated by airplanes.

Example 2. $\forall x, x < x^2$. Is this true or false? How we fix it if we should?

These two quantifiers (plus the usual logical operations such as conjunction and disjunction, i.e. AND, OR,...) are sufficient to write all statements in mathematics. This gives rise to a standard mathematical language, which greatly facilitates expressing mathematical reasoning in proofs and problem solving, which we will be using throughout this course.

<u>Negation with Quantifiers. Predicate Negation Laws.</u>

Predicate Negation Laws. [Generalized De Morgan]

$\sim(\exists x \in X: p_i) \equiv \forall x \in X: \sim p_i$

$\sim(\forall x \in X: p_i) \equiv \exists x \in X: \sim p_i$

Negation of statements with quantifiers and implications.

1. $(\exists x \in X: P(x)) \equiv$ there exists x in X such that $P(x)$ is satisfied. The negation of it would be,

$\sim(\exists x \in X: P(x)) \equiv$ (It is not the case that there exists x in X such that $P(x)$ is satisfied) $\equiv$ (for any x in X opposite of $P(x)$ is satisfied) $\equiv$ $(\forall x \in X: \sim P(x))$.

2. $(\forall x \in X: P(x))\equiv$ (for any $x \in X$ $P(x)$ is satisfied). Negation of it would be,

$\sim(\forall x \in X: P(x)) \equiv$ (It is not the case that for any x in X $P(x)$ is satisfied) $\equiv$ (there exists x in X such that $P(x)$ is not satisfied) $\equiv$ $(\exists x \in X: \sim P(x))$.

Example 1. The negation of a proposition (there are positive integers n such that $2^{2^n} + 1$ is not a prime) would be a proposition, (for every positive integer n, $2^{2^n} + 1$ is a prime),

$\sim(\exists n \in$ N: $2^{2^n} + 1$ is not a prime) $\equiv$ ($\forall n \in N$: $2^{2^n} + 1$ is a prime).

Example 2. The negation of a proposition (Every prime is odd) would be a proposition that not every prime is odd, or, that there exists at least one prime that is even,

$\sim(\forall n$, (n is prime): (n is odd)) $\equiv$ ($\exists n$, (n is prime): (n is even)).

In fact, even a stronger proposition holds, $(\exists!\, n, (n \text{ is prime}): (n \text{ is even}))$.

Negation with Multiple Quantifiers.

3. $((\forall x \in X), (\exists y \in Y): P(x,y)) \equiv$ (for all x in X there exists y in Y such that $P(x,y)$ is satisfied). The negation of it would be,

$$\sim\big((\forall x \in X), (\exists y \in Y): P(x,y)\big) \equiv \big((\exists x \in X), (\forall y \in Y): \sim P(x,y)\big)$$

Negation of Implications and Equivalencies.

1. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$

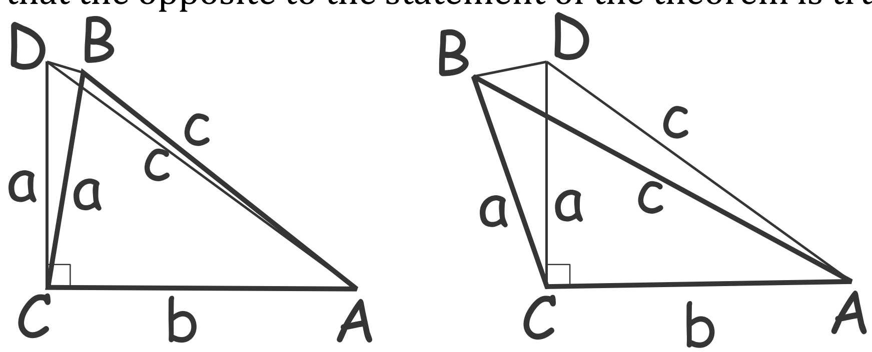
The negation of (A implies B) $\equiv$ (B follows from A) would be a proposition that a conjunction of A and $\sim(B)$ holds, (both A and an opposite of B hold).

2. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
3. $\sim(A \Leftrightarrow B) \Leftrightarrow \big(A \Leftrightarrow \sim(B)\big)$

The negation of (A and B are equivalent) would be (A and opposite of B are equivalent), or, (opposite of B and A are equivalent).

Example. The inverse Pythagorean theorem. ($\forall$ triangle ABC with sides a, b, and c, $a^2 + b^2 = c^2$) $\Rightarrow$(ABC is a right triangle with hypothenuse c and legs a, b).

Proof . Proof by contradiction (*reductio ad absurdum*) proceeds by assuming that the opposite to the statement of the theorem is true,



$\sim$(($\forall$ triangle ABC with sides a, b, and c, $a^2 + b^2 = c^2$)$\Rightarrow$(ABC is a right triangle with hypothenuse c and legs a, b)), or,

($\exists$ triangle ABC with sides a, b, and c, $a^2 + b^2 = c^2$)$\wedge$(ABC is not a right triangle)

One way to obtain the contradiction is illustrated by the auxiliary additional construction shown below (consider the angles $\widehat{CDB} = \widehat{CBD} > \widehat{ADB} = \widehat{ABD}$). There are also other ways.

Exercise. What other proofs can you suggest?

Recap. A summary of logical equivalences.

Commutative laws:

1. $(A \wedge B) \Leftrightarrow (B \wedge A)$
2. $(A \vee B) \Leftrightarrow (B \vee A)$
3. $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$

Associative laws:

1. $\big(A \wedge (B \wedge C)\big) \Leftrightarrow \big((A \wedge B) \wedge C\big)$
2. $\big(A \vee (B \vee C)\big) \Leftrightarrow \big((A \vee B) \vee C\big)$
3. $\big(A \Leftrightarrow (B \Leftrightarrow C)\big) \Leftrightarrow \big((A \Leftrightarrow B) \Leftrightarrow C\big)$

Distributive laws:

4. $\big(A \wedge (B \vee C)\big) \Leftrightarrow \big((A \wedge B) \vee (A \wedge C)\big)$
5. $\big(A \vee (B \wedge C)\big) \Leftrightarrow \big((A \vee B) \wedge (A \vee C)\big)$
6. $\big(A \Rightarrow (B \wedge C)\big) \Leftrightarrow \big((A \Rightarrow B) \wedge (A \Rightarrow C)\big)$
7. $\big(A \Rightarrow (B \vee C)\big) \Leftrightarrow \big((A \Rightarrow B) \vee (A \Rightarrow C)\big)$
8. $\big((A \wedge B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \vee (B \Rightarrow C)\big)$
9. $\big((A \vee B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \wedge (B \Rightarrow C)\big)$

Negation laws:

4. $\sim(A \wedge B) \Leftrightarrow \big(\sim(A) \vee \sim(B)\big)$
5. $\sim(A \vee B) \Leftrightarrow \big(\sim(A) \wedge \sim(B)\big)$
6. $\sim(\sim A) \Leftrightarrow A$
7. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$

8. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
9. $\sim(A \Leftrightarrow B) \Leftrightarrow \left(A \Leftrightarrow \sim(B)\right)$

Implication laws:

1. $(A \Rightarrow B) \Leftrightarrow \left(\sim\left(A \wedge \sim(B)\right)\right)$
2. $(A \Rightarrow B) \Leftrightarrow (\sim(A) \vee B)$
3. $(A \Rightarrow B) \Leftrightarrow \left(\sim(B) \Rightarrow \sim(A)\right)$
4. $(A \Leftrightarrow B) \Leftrightarrow \left((A \Rightarrow B) \wedge (B \Rightarrow A)\right)$
5. $(A \Leftrightarrow B) \Leftrightarrow \left(\sim(A) \Leftrightarrow \sim(B)\right)$

Recap. Properties of rational numbers ($\mathbb{Q}$) and algebraic operations.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{Q}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in \mathbb{Q}, \exists c \in \mathbb{Q}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{Q}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{Q}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in \mathbb{Q}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{Q}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{Q}, \exists 0 \in \mathbb{Q}, a + 0 = a$
- $\forall\, a \in \mathbb{Q}, \exists - a \in \mathbb{Q}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{Q}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{Q}, (a < b) \Rightarrow (a + c < b + c)$

Multiplication and division.

- $\forall\, a, b \in \mathbb{Q}, a \cdot b = b \cdot a$
- $\forall\, a, b, c \in \mathbb{Q}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $\forall\, a, b, c \in \mathbb{Q}, (a + b) \cdot c = a \cdot c + b \cdot c$
- $\forall\, a \in \mathbb{Q}, \exists 1 \in \mathbb{Q}\,, a \cdot 1 = a$
- $\forall\, a \in \mathbb{Q}, a \neq 0, \exists \frac{1}{a} \in \mathbb{Q}, a \cdot \frac{1}{a} = 1$
- $\forall\, a, b \in \mathbb{Q}, b \neq 0, \frac{a}{b} = a \cdot \frac{1}{b}$
- $\forall\, a, b, c \in \mathbb{Q}$, c>0, $(a < b) \Rightarrow (a \cdot c < b \cdot c)$
- $\forall\, a \in \mathbb{Q}, a \cdot 0 = 0, a \cdot (-1) = -a$