MATH 8: HANDOUT 15 [JAN 26, 2025]
EUCLIDEAN GEOMETRY 5: TRAPEZOID. CENTROID OF A TRIANGLE.

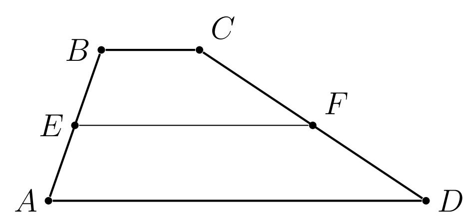
12. Special quadrilaterals : trapezoid

Today we continue the discussion of quadrilaterals with a trapezoid.

Definition. A quadrilateral is called a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair may not be.

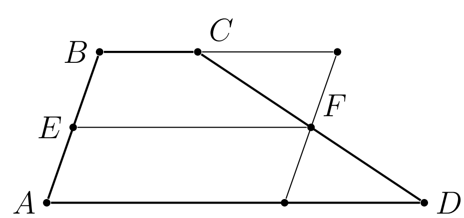
If the other two sides are also parallel, then it becomes a parallelogram, so all theorems that apply to a trapezoid will also apply to a parallelogram, although some may become trivial. The most interesting property of a trapezoid is its midline:

Definition. The midline of a trapezoid $ABCD$ ($AD \parallel BC$) is the segment connecting the midpoints of its sides (AB and CD).



Theorem 20. *[Trapezoid midline] Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively. Then $\overline{EF} \parallel \overline{AB}$, and $EF = (AD + BC)/2$.*

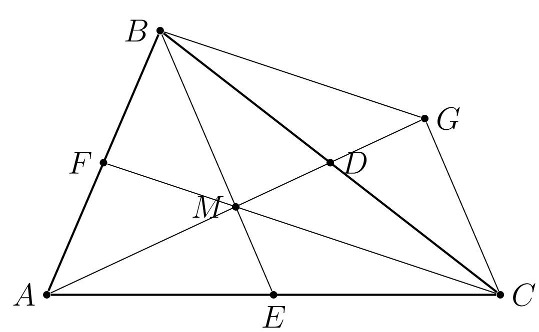
Idea of the proof: Draw through point F a line parallel to AB, as shown in the figure. Prove that this gives a parallelogram, in which points E, F are midpoints of opposite sides.



Of course, the above theorem is automatically fulfilled for a parallelogram, and the midline will be congruent to the sides it is parallel to.

13. Intersection point of medians

Theorem 21. *[Intersection point of medians in a triangle] Let ABC be a triangle and AD, BE, and CF are its medians. Then AD, BE, and CF intersect at a single point M and each is divided by it $2:1$ counting from their respective vertices: $AM : MD = BM : ME = CM : MF = 2:1$.*



First, let's prove that if BE and CF are medians intersecting at point M, and AM is extended to intersect BC at the point D, then AD is also a median.

Proof. Continue line AD beyond poind D and mark point G such that $GM = AM$.

1. M is the midpoint of AG, and E is the midpoint of AC; therefore ME is a midline of $\triangle AGC$ and $ME \parallel GC$;
2. similarly, MF is a midline of $\triangle AGB$ and $MF \parallel GB$;

3. from the above, $BMCG$ is a parallelogram, and its diagonals BC and MG bisect each other, so D is the midpoint of BC and AD is a median.

$\square$

Proving that $|AM| = 2|MD|$ and also $|BM| = 2|ME|$, $|CM| = 2|MF|$ is left as homework. By now, we know that the following lines in any triangle intersect at the same point:

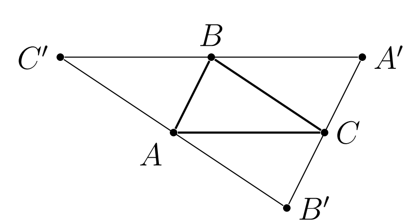
- the three angle bisectors intersect at the same point (*incenter*), which is *equidistant from the three sides* of the triangle; it is inside the triangle.
- the three perpendicular side bisectors intersect at the same point (*circumcenter*), which is *equidistant from the three vertices (corners)* of the triangle; it may be inside or outside the triangle.
- the three altitudes intersect at the same point, which is called the *orthocenter*, and may be *inside or outside the triangle*;
- and the three medians intersect at the same point, which is called the *centroid*, and are divided by it 2:1 counting from the triangle vertices. The centroid is inside the triangle.

The centroid of a triangle (intersection point of the medians) has a remarkable property: it is a center of mass of a uniform triangle. You can check this by cutting out a triangle from a sheet of cardboard or other uniform material and balancing it on the tip of a needle. The same point will also be the center of mass if you place *three equal masses* at each vertex.

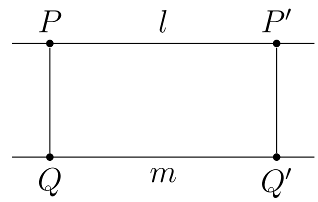
HOMEWORK

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

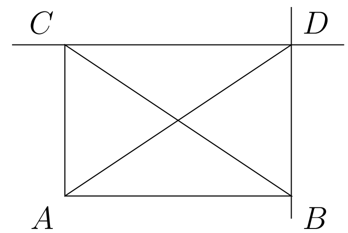
1. Finish the proof of Theorem 20: show that the length of the midline $EF = (AD + BC)/2$.
2. Finish the proof of Theorem 21: show that the intersection point splits medians $2:1$ counting from the vertex.
3. Review the proof that the three altitudes of a triangie intersect at a single point: Given a triangle $\triangle ABC$, draw through each vertex a line parallel to the opposite side. Denote the intersection points of these lines by A', B', C' as shown in the figure.
   (a) Prove that $A'B = AC$ (hint: use parallelograms!)
   (b) Show that B is the midpoint of $A'C'$, and similarly for other two vertices.
   (c) Show that altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of sides of $\triangle A'B'c'$.
   (d) Prove that the three altitudes of $\triangle ABC$ intersect at a single point.



4. (Distance between parallel lines)
Let l, m be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $\overleftrightarrow{PQ} \perp l$ (by Theorem 6, this implies that $\overleftrightarrow{PQ} \perp m$)). Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $\overleftrightarrow{P'Q'} \perp l$, we have $PQ = P'Q'$. (This common distance is called the distance between l, m.)



5. Let $\triangle ABC$ be a right triangle ($\angle A = 90°$), and let D be the intersection of the line parallel to $\overline{AB}$ through C with the line parallel to $\overline{AC}$ through B.



(a) Prove $\triangle ABC \cong \triangle DCB$
(b) Prove $\triangle ABC \cong \triangle BDA$
(c) Prove that $\overline{AD}$ is a median of $\triangle ABC$.

6. Let $\triangle ABC$ be a right triangle ($\angle A = 90°$), and let D be the midpoint of $\overline{BC}$. Prove that $AD = \frac{1}{2}BC$.