MATH 8B [JAN 19, 2024]
HANDOUT 14: EUCLIDEAN GEOMETRY 4: QUADRILATERALS. MIDLINE OF A TRIANGLE.

10. Special quadrilaterals

In general, a figure with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$ etc.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

Definition. A quadrilateral is called

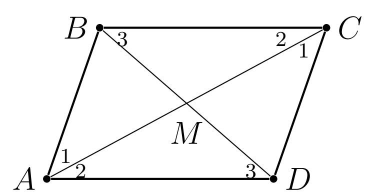
- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have a number of useful properties.

Theorem 16. *Let $ABCD$ be a parallelogram. Then*

- $AB = DC$, $AD = BC$
- $m\angle A = m\angle C$, $m\angle B = m\angle D$
- *The intersection point M of diagonals AC and BD bisects each of them.*

Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Thus, by ASA, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly one proves that $m\angle A = m\angle C$ (or note in the diagram that both are $\angle 1 + \angle 2$).



Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC$, $BM = MD$. $\square$

There are several ways you can recognize a parallelogram; in fact, conclusions in Theorem 16 are not only necessary, but also sufficient for a quadrilateral to be a paralellogram. Let's remember them as a single theorem:

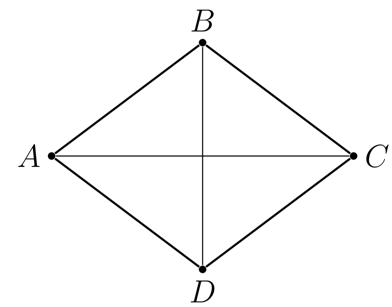
Theorem 17. *Any quadrilateral $ABCD$ is a parallelogram if any one of the following conditions is true:*

- *its opposite sides are equal ($AB = CD$ and $AD = BC$),* ***OR***
- *two opposite sides are equal and parallel ($AB = CD$ and $AB \parallel CD$),* ***OR***

- *its diagonals bisect each other ($AM = CM$* and *$BM = DM$, where $AC \cap BD = M$),* ***OR***
- *its opposing angles are equal ($\angle BAD = \angle BCD$* and *$\angle ABC = \angle ADC$).*

Proofs are left to you as a homework exercise.

Theorem 18. *Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.*

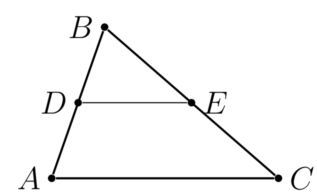


Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem 17 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 12, it is also the altitude. □

11. Midline of a triangle

Properties of parallelograms are very useful for proving theorems, for example about a triangle midline.

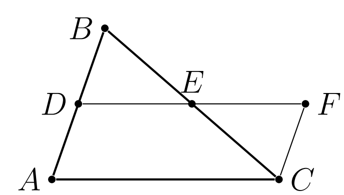
Definition. A midline of a triangle $\triangle ABC$ is the segment connecting midpoints of two sides.



Theorem 19. *If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $\overline{DE} \parallel \overline{AC}$.*

Proof. Continue line DE and mark on it point F such that $DE = EF$.

1. Since BC and DF bisect each other at their intersection, $BDCF$ is a parallelogram. So $BD = FC$ and $BD \parallel FC$.
2. We deduce $AD \parallel CF$ (since ADB is a straight line and $BD \parallel FC$. Also since D is the midpoint of side AB, we have $AD = BD = CF$. Therefore $ADFC$ is a parallelogram.
3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$.



□

Alternatively, one can prove that if a line parallel to one side of the triangle crosses another side in the middle, then it is a midline, and will cross the third side also in the middle.

Homework

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. Prove that in a parallelogram, sum of two adjacent angles is equal to $180°$:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°$$

2. **[We may have done some in class - you still need to write the proofs neatly]** Prove Theorem 17: that a quadrilateral is a parallelogram if

(a) it has two pairs of equal sides;
(b) if two of its sides are equal and paraallel;
(c) if its diagonals bisect each other;
(d) if its opposite angles are equal.

Any of the above statements can be used as the definition of a parallelogram.

3. (Rectangle) A quadrilateral is called rectangle if all angles have measure $90°$.

(a) Show that each rectangle is a parallelogram.
(b) Show that opposite sides of a rectangle are congruent.
(c) Prove that the diagonals of a rectangle are congruent.
(d) Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.

4. Is there any relationship between the angles of a trapezoid?

5. Show that in any triangle, its three midlines divide the original triangle into four triangles, all congruent to each other.

***6.** Prove that in any triangle, its altitudes intersect at the same point.
Hint: consider a triangle and its three midlines from the previous problem; draw the perpendicular side bisectors to each side of the big triangle. Are they altitudes in some other triangle?