MATH 8B [2024 NOV 10]
HANDOUT 9 : LOGIC 4 : PROOFS

We are commonly asked to *prove* something: given a series of statements (assumptions), prove another statement (conclusion).

In the simple case where all the statements are just formulas involving some logic variables and basic operations, one way to do it is by writing a truth table: list all possible combinations of values of variables and verify that in each case where all assumptions are true, the conclusion is also true. However, usually that's not how it is done. Instead, we construct a series of intermediate statements, each of which follows from the previous ones and assumptions.

When constructing these intermediate statements, we can again use truth tables or common logic laws:

- Given $A \implies B$ and A, we can conclude B *(Modus Ponens)*
- Given $A \implies B$ and $B \implies C$, we can conclude that $A \implies C$. [Note: it doesn't mean that in this situation, C is always true! It only means that **if** A is true, then so is C.]
- Given $A \wedge B$, we can conclude A (and we can also conclude B)
- Given $A \vee B$ and $\neg B$, we can conclude A
- Given $A \implies B$ and $\neg B$, we can conclude $\neg A$ *(Modus Tollens)*
- $\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$ *(De Morgan Law)*
- $(A \implies B) \iff ((\neg B) \implies (\neg A))$ *(Law of contrapositive)*
- [Proof by cases] Given $A \vee B$, $A \implies C$ and $B \implies C$, we can conclude C

Note: it is important to realize that statements $A \implies B$ and $B \implies A$ are **not** equivalent! (They are called converse of each other).

COMMON METHODS OF PROOF

Proof by cases.

Example: Prove that for any integer n, the number $n(n+1)$ is even.

Proof. If n is integer, it is even or odd. If n is even, then $n(n+1)$ is even (a multiple of even is always even). If n is odd, then $n+1$ is even and thus $n(n+1)$ is even too. Thus, in all cases $n(n+1)$ is even. □

General scheme:

Given

$$A_1 \vee A_2$$
$$A_1 \implies B$$
$$A_2 \implies B$$

we can conclude that B is true.

You can have more than two cases.

Note: it is important to verify that the cases you consider cover all possibilities (i.e. that at least one of the statements A_1, A_2 is always true).

Conditional proof.

Example: Prove that if n is even, then n^2 is even.

Proof. Assume that n is even. Then $n^2 = n * n$ is also even, since a multiple of even is even. □

General scheme

To prove $A \implies B$, we can

- Assume A
- Give a proof of B (in the proof, we can use that A is true).

This proves $A \implies B$ (without any assumptions).

Proof by contradiction.

Example: Prove that if x is a real root of polynomial $p(x) = 10x^3 + 2x + 15$, then x must be negative.

Proof. Assume that x is not negative, i.e. $x \geq 0$. Then $p(x) = 10x^3 + 2x + 15 \geq 15$, which contradicts the fact that x is a root of $p(x)$. Thus, our assumption can not be true, so x must be negative. □

General scheme

To prove that A is true, assume A is false, and derive a contradiction. This proves that A must be true.

Necessary and sufficient conditions. The implication $A \implies B$ is sometimes called "A is a sufficient condition for B". In other words, if A is true then that is sufficient for B to be true.

It can also be phrased as "B is a necessary condition for A". In other words, for A to be true it is necessary that B is true.

Note that "A is sufficient for B" (i.e., $A \implies B$) is converse to "A is necessary for B" (i.e., $B \implies A$).

PROBLEMS

In problems 1, 2, you need to (a) write the obvious conclusion from given statements; and (b) justify the conclusion, by writing a chain of arguments which leads to it. It may help to write the given statements and conclusion by logical formulas (denoting the statements which are used by letters $A, B, \dots$ connected by logical operations $\vee, \wedge, \implies, \dots$).

1. If today is Thursday, then Jane's class has library day. If Jane's class has library day, then Jane will bring home new library books. Jane brought no new library books. Therefore,...

2. If Jack comes home late from school, it means he either had a track meet or a theater club. After a track meet, he comes home very tired. Today he came home late but was not tired. Therefore, ...

3. Consider the following statement:

You can't be happy unless you have a clear conscience.

Can you rewrite it using the usual logic operations such as $\wedge$, $\vee$, $\implies$? Use letter H for "you are happy" and C for "you have a clear conscience".

4. You probably know Lewis Carroll as the author of *Alice in Wonderland* and other books. What you might not know is that he was also a mathematician very much interested in logic, and had invented a number of logic puzzles. Here is one of them:

You are given 3 statements.

(a) All babies are illogical.
(b) Nobody is despised who can manage a crocodile.
(c) Illogical persons are despised.

Can you guess what would be the natural conclusion from these 3 statements? Can you prove it using some laws of logic?

It might help to write each of them as combination of elementary statements about a given person, e.g. B for "this person is a baby", I for "this person is illogical", etc.

5. Here is another of Lewis Carrol's puzzles.

(a) All hummingbirds are richly colored.
(b) No large birds live on honey.
(c) Birds that do not live on honey are dull in color.

Therefore, ...

(You may assume that "dull in color" is the same as "not richly colored"). Hint: think of all these as statements about some bird X and rewrite in simpler form, using only basic logic operations. E.g., first statement can be rewritten as "If X is a hummingbird, then X is richly colored".

6. Prove by contradiction that there does not exist a smallest positive real number.

7. Use proof by contradiction to prove the following statement:

If the square of an integer number n is even, then n itself is even.

You can use without proof that every integer number is either even (i.e., can be written in the form $n = 2k$, with integer k) or odd (i.e., can be written as $n = 2k+1$, with integer k).

8. Let A, B, C be logical variables. Given that the following are true:

$$A \vee B$$
$$B \implies \neg C$$
$$C \implies ((\neg A) \vee B)$$

prove that C is false.

9. Show that $a < b$ is a sufficient condition for $4ab < (a+b)^2$? Is this condition also necessary?

10. Prove the following logical equivalence $\neg(p \implies q) \iff (p \wedge \neg q)$.

11. Given logical statements m, p, q, let a denote the combined statement $m \wedge p) \vee (\neg m \wedge q)$. Prove the following:

(a) If m is true, then $a \iff p$
(b) If m is false, then $a \iff p$