MATH 8B [10/20/2024]
HANDOUT 5: MORE COMBINATORICS, REVIEW

1. How many ways are there to group $2n$ people into n pairs? [Pairs are not numbered.]

2. How many diagonals are there in a convex octagon?

3. If n distinct dice are throwm simultaneously, how many different choices of numbers facing up are possible?

4. If n indistinguishable dice are thrown simultaneously, how many possible combinations of numbers facing up are possible?

5. When two dice are thrown simultaneously, what is the sum you are likeliest to get, and what is its probability?

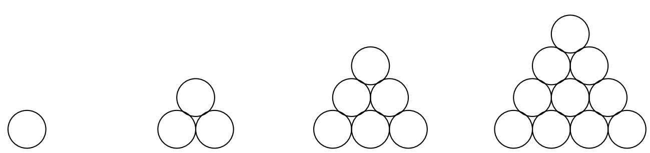
6. Deduce that Pascal's triangle is symmetric, i.e. $\binom{n}{k} = \binom{n}{n-k}$ in three ways:
- (a) From the definition (choosing k objects out of n distinct objects).
- (b) Using the formula.
- (c) Using the binomial theorem for $(x+y)^n$ and $(y+x)^n$.

7. Show that $(6!)!$ is a multiple of $(6!)^{5!}$. [Hint: Consider an alphabet with $5! = 120$ elements, and find out how many words you can make from a collection where each letter is repeated 6 times.]

***8.** A frog on vacation to India is attempting to climb a step at the Taj Mahal. The frog, figuring it has a one in ten chance to succeed at the jump, decides to attempt the jump ten times (if it succeeds early, it won't make further attempts). What is the chance that the frog will make it up the step in its ten attempts?

9. If you have 5 lines on the plane so that no two are parallel and there are no triple intersection points, how many triangles do they form? How many closed polygons do they form? What if there are n lines?

10. Let T_n be the number of circles in a triangular shape with n levels like the ones below (these are sometimes called *triangular* numbers):



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$

- (a) Note that $T_3 = T_2 + 3$, and $T_4 = T_3 + 4$. Is it true that in general, $T_n = T_{n-1} + n$? Why or why not?
- (b) Look at Pascal's triangle. Can you find these numbers there?
- (c) Can you write a general formula for T_n?

***11.**
- (a) What if instead of drawing circles on plane, we were arranging balls in a (square) pyramid? Can you guess how many balls we would have in pyramid with 1 level; with 2, 3, 4 levels?
- (b) What if we were arranging balls in a tetrahedron?

12. How many ways are there of choose n books from a total $2n$ books, which are arranged on two bookshelves with n books each? By counting the number in two ways, show that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

***13.** (a) There are n distinct items arranged in a sequence. A subset of these n items is called unfriendly if no two of its elements are consecutive in the sequence. How many unfriendly subsets of k elements are there? [Hint: think of the chosen elements are red marbles, and the non-chosen elements as white marbles.]

(b) How many unfriendly subsets (of any size) are there? [Hint: this is essentially the problem of n coin tosses, with no two consecutive heads. Think of "heads" as the item being part of the subset and "tails" as not being part of the subset.]

***14.** Show that the n'th Fibonacci number (where $F_0 = 1$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$) also satisfies the identity

$$F_{n+1} = \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots \binom{n-n_0+1}{n_0}$$

where $n_0 = \lfloor \frac{n+1}{2} \rfloor$.