MATH 8B [09/22/2024]
HANDOUT 2: PASCAL'S TRIANGLE

Note that every entry in this triangle is obtained as the sum of two entries above it.

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&&\\
&&&&&&&1&&1&&&&&&&\\
&&&&&&1&&2&&1&&&&&&\\
&&&&&1&&3&&3&&1&&&&&\\
&&&&1&&4&&6&&4&&1&&&&\\
&&&1&&5&&10&&10&&5&&1&&&\\
&&1&&6&&15&&20&&15&&6&&1&&\\
&1&&7&&21&&35&&35&&21&&7&&1&\\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

The k-th entry in n-th line is denoted by $\binom{n}{k}$, or by ${}_nC_k$. Note that both n and k are counted from 0, not from 1: for example, $\binom{2}{1} = 2$. Thus, these numbers are defined by these rules:

$$\begin{aligned}
&\binom{n}{0} = \binom{n}{n} = 1\\
&\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{for } 1 \leq k \leq n-1
\end{aligned} \tag{1}$$

We can derive the second (recursive) formula as follows. Suppose I have a collection of n books, and from these I must choose k books to place on my coffee table. Let $M(n,k)$ refer to the number of ways there are to choose this subset of k books.

Suppose one of my n books is H : "The Hobbit". Let a be the number of ways to choose k books, in which H is excluded in the coffee-table collection, and b the number of ways in which H is included.

1. Prove that $a + b = M(n,k)$.
2. Prove that $a = M(n-1,k)$, and $b = M(n-1,k-1)$.
3. Deduce from the above that $M(n,k) = \binom{n}{k}$, i.e.

Therefore the rule,

Number of way to choose k persons out of n (order doesn't matter)$= \binom{n}{k}$

Remember, there is an explicit formula for binomial coefficients:

$${}_nC_k = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

You can directly check both properties above from this formula (problem 2). Conversely, if you did not know the formula, but just the recursion properties (1), you would be able to figure out the explicit formula using the principle of mathematical induction (which you will learn next year).

PROBLEMS

General note: In this homework assignment (and in all other assignments in this class), many problems are non-trivial and require some thought. Try to start early. You are not expected to be able to solve all of the problems, so do not be discouraged if you can't solve some of them. Please make sure that you show not just the answer but also the solution, i.e. your reasoning showing how you arrived to this answer. Ideally, your solution should be such that someone who doesn't know how to solve this problem can read it and follow your arguments.

It is enough if you can write the answers in terms of factorials and binomial coefficients — it is not necessary to actually compute them: an answer like $13!$ or $\binom{10}{5}$ is good enough.

1. If we want to choose a president, vice-president, and two assistants from a 15-member club, in how many ways can we do it?

2. Verify the relations (1) hold by plugging in the formulas for the binomial coeffieicients.

3. 5 kids come to a store to choose Halloween costumes. The store sells 25 different costumes. Assuming the store has enough stock for the kids to choose the same costume if they want, in how many ways can the kids choose the costumes? What if they want to choose so that all costumes are different?

4. Suppose I flip a coin three times, and I record its result each time (for example, the coin may land heads then tails then heads, which I will write as HTH, where order matters). I will refer to this three letter combination as the final result - for example, HHH is the only final result that has no tails.

(a) How many final results are there with exactly one tail?

(b) How many final results are there with exactly two tails?

5. Five octopuses are working at the beach's local landfood restaurant. They want to assign lunch shifts, so that some of the octopuses can have lunch from 12:00pm to 1:00pm, and the others can have lunch from 1:00pm to 2:00pm.

(a) If they decide to have two octopuses take the first shift and three take the second, how many possible ways are there to assign shifts?

(b) If the noontime hour is especially busy and they decide to have just one octopus take the first shift and the remaining four take the second shift, how many ways are there to assign shifts?

6. Are there any rows in the Pascal triangle where all numbers are odd? Which rows are they? Can you prove your answer? (Hint: use the first form of the explicit formula for binomial coefficients.)

7. Are there any rows in Pascal's triangle where all the numbers (except the 1's at the end) are even? Which rows are they? Can you prove your answer? (Hint: use the previous problem.)

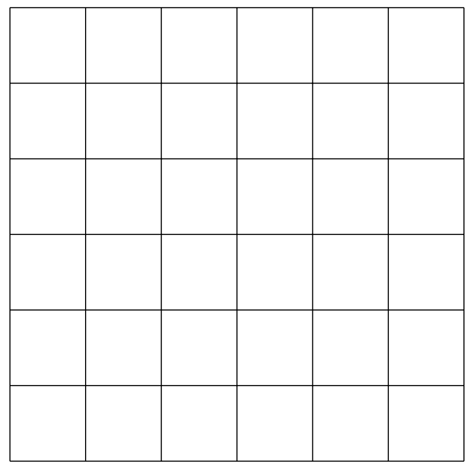
8. What is the sum of all numbers in n-th row of Pascal triangle? Can you guess the pattern — and once you guessed it, justify your guess.

9. How many ways are there to place two rooks on a chessboard so that they are not attacking each other? [For those of you who are unfamiliar, a chessboard is an 8×8 grid of squares, and *rooks* are pieces that can occupy any one of these individual squares, and may attack any other piece that's in the same row or column of the board as itself.]

10. How many different paths are there on 6×6 chessboard connecting the lower left corner with the upper right corner such that

- The path always goes to the right or up, never to the left or down.
- The path never goes below the diagonal (being on diagonal is OK)

Hint: each cell can be reached from the ones to the left (except the first column) or below (except the diagonal). Count the total number of paths leading to each accessible cell.



11. Suppose I flip a coin 12 times, and record the sequence of results (for example, HTHTHTHHTTHH). How many sequences of coin tosses are there for which no two consecutive tosses are heads (i.e., HH does not appear)? [Hint: let $f(n)$ be the number of such sequences when there are n coin tosses, and write a recursion for $f(n)$.]