March 2, 2025 Math 8

Handout 21. Euclidean Geometry 10: Construction with ruler and compass.

Construction with ruler and compass.

Sometimes we need to construct a figure with specified sides, angles, or other geometric features. Imagine staking the ground to mark the boundaries of foundation for a new house, or an industrial building of an arbitrary shape. We would proceed in mostly the same manner as on paper, using line-of-sight and thread instead of a straightedge and a ruler instead of a compass.

It is sufficient to know how to construct the following, using only a straightedge and a compass:

- construct a segment equal to another
- construct an angle equal to another
- bisect a segment
- bisect an angle
- construct a line perpendicular to another
- construct a line parallel to another.

Combining these elements, one can construct triangles, quadrilaterals, and other figures based only on description and measures. However, this is only the first part of the solution; proving that the figure constructed is unique is the second part. Sometimes the result is not unique, so we have to construct all possible solutions.

To find all solutions, it is useful to think of sets of all points (commonly, a line, a line segment, or a curve), whose location satisfies or is determined by one or more specified conditions. Such a set is called a "locus" (plural "loci"). Recall the center of the circle circumscribed around 4ABC: it is the intersection of loci,

- equidistant from vertices A and B (perp. bisector to side AB)
- equidistant from vertices A and C (perp. bisector to side AC)
- equidistant from vertices B and C (perp. bisector to side BC)

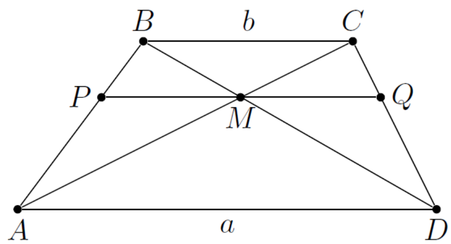
Since all these loci have a common point, a solution exists; because it is a single point, the solution is unique. Other useful loci are,

- bisector of angle $\angle AOB$: locus of points equidistant from lines (OA), (OB)
- line l parallel to line a: locus of points at the same fixed distance from a, in the same half-plane;
- circle $\omega(O;R)$: locus if points at distance R from a single point O;
- arc $\widehat{A\alpha B}$ of circle $\omega(O;R)$: locus of points C such that $\angle ACB = \frac{1}{2}\angle AOB$.

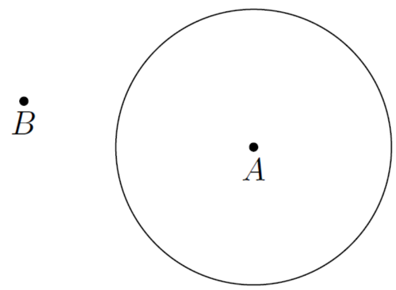
Homework problems

This homework may be more challenging than usual. Try to solve as many problems as you can, and we will discuss them all in class.

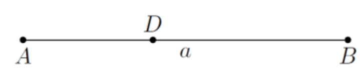
1. Consider the trapezoid with bases $|AD| = a$, $|BC| = b$. Let M be the intersection point of diagonals and let PQ be the segment parallel to the bases through M.
   a. Show that point M divides each of diagonals in proportion $a{:}b$, e.g. $|AM|{:}|MC| = a{:}b$.
   b. Show that points P, Q divide sides of the trapezoid in proportion $a{:}b$.
   c. Show that $|PQ| = \frac{2ab}{a+b}$. [Hint: compute $|PM|$, $|MQ|$ separately and then add].



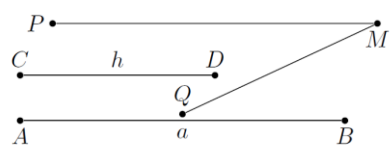
2. In a $\triangle ABC$, with all angles acute, AD, BE, and CF are altitudes. Find angles of $\triangle DEF$.
3. Given a circle ω with center A and a point B outside this circle, construct the tangent line l from B to ω using straightedge and compass. How many solutions does this problem have?



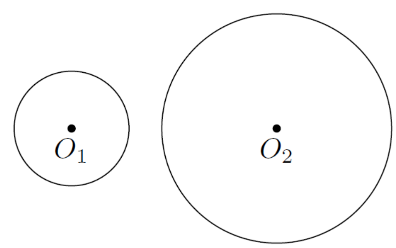
4. Given a segment $|AB| = a$ and point $D \in [AB]$, construct a right triangle $\triangle ABC$ with hypotenuse $|AB| = a$ and CD being
   a. a bisector of the right angle $\angle ABC$
   b. an altitude from vertex C to the side AB.



5. Construct triangle $\triangle ABC$ such that $|AB| = a$, $\angle ABC = \angle PMQ$, and altitude from vertex C to side AB is $|CD| = h$. Is there only one such triangle? [Hint: point C is at distance h from line (AB), and points A and B are visible from point C at angle $\angle PMQ$. Which loci does it correspond to]?



6. Given two circles with centers O_1; O_2 and radiuses r_1; r_2, respectively, construct (using straightedge and compass) a common tangent line to these circles. You can assume that circles do not intersect: $|O_1O_2| > r_1 + r_2$ and that $r_2 > r_1$. [Hint: assume that we have such a tangent line, call it l. Then distance from that line to O_1 is r_1, and distance to O_2 is r_2. Thus, if we draw a line l_0 parallel to l but going through O_1, the distance from l_0 to O_2 is . . . and thus l_0 is tangent to ...].



7. Describe how you would construct a triangle $\triangle ABC$ with given side $|BC| = a$, acute angle $\angle ABC = \alpha$ and
   a. the sum of sides, $|AB| + |AC| = d > a$
   b. the difference of sides, $|AB| - |AC| = e < a$