January 26, 2025 Math 8

# Handout 18. Euclidean Geometry 7: Circles. Inscribed and circumscribed quadrilaterals.

## Quadrilaterals and circumscribed circles.

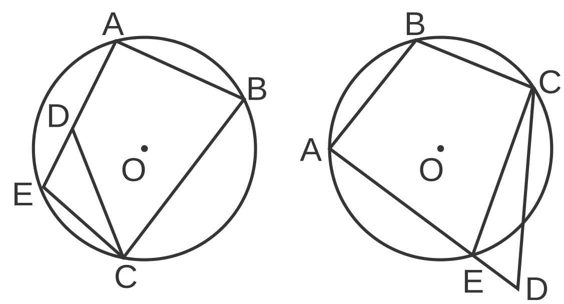
In the previous class we discussed the remarkable relation between the inscribed and central angles in any circle. From this, we can easily prove the following property of an inscribed quadrilateral.

**Theorem 27.** If a quadrilateral $ABCD$ can be circumscribed by a circle, then the sum of two opposite angles is equal to 180°: $\angle ABC + \angle ADC = 180° = \angle BAD + \angle BCD$.

The opposite is also true, i.e. the condition for the angles above is also sufficient that the quadrilateral $ABCD$ can be circumscribed by a circle. This can be proven using the fact that an arc of a circle is a special locus of points:

**Theorem 28.** For any segment $AB$, the locus of points $C$ such that $\angle ABC = \alpha$ is the arc of a circle $\omega(O; |OA| = |OB|)$, for which $AB$ is a chord, such that $\angle AOB = 2\alpha$.

Note that there are two arcs spanned by the same circle, and the inscribed angles with vertices on these arcs are supplementary (add up to 180°).




**Proof**. Indeed, any angle with vertex $D$ inside the circle will be larger than $\alpha$, and any angle with vertex outside the circle will be smaller than $\alpha$. In the first case, this can be shown by continuing one side of the angle to intersection with the circle ($C$) and considering $\triangle ACD$; the second case is analogous. □

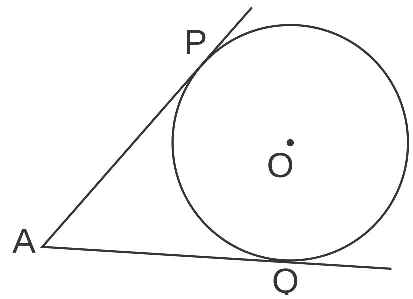
Application of the same argument to quadrilaterals inscribed into a circle is shown in the Figure.

## Quadrilaterals with inscribed circles.

**Definition**. A circle inscribed into a triangle, quadrilateral, or other polygon (if it exists) is tangential to all the sides.

The radius drawn from the center to the point common with a particular side is perpendicular to it, and is thus equal to the distance from the center to the side. The center is therefore equidistant from all the sides. We already know that one can always inscribe a circle into a triangle: the center of the inscribed circle is the common intersection point of the three angle bisectors. However, a circle can be inscribed only into special quadrilaterals. Similar to the case of a triangle, the in-center of any other polygon must lie on the intersection of angle bisectors. However, in general case these bisectors are not concurrent (do not intersect at a single common point). In order to check whether in a quadrilateral all four bisectors intersect at the same point ("concur"), we can use a theorem about tangential segments to a circle drawn from an outside point:

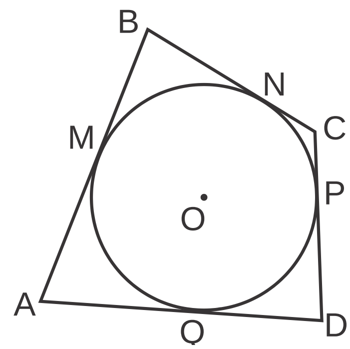
**Theorem 29.** Let $A$ be a point outside a circle $\omega(O; R)$ and $AP$, $AQ$ are tangential to it $P, Q \in \omega$. Then, $|AP| = |AQ|$.




**Proof**. The proof is straightforward and left as an exercise. □

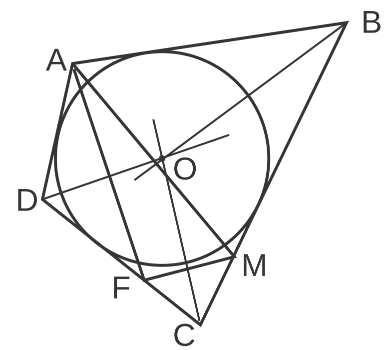
The following corollary of this theorem establishes a necessary and sufficient condition for a circle to be inscribable in a quadrilateral.

**Theorem 30.** A circle can be inscribed into a quadrilateral $ABCD$ if and only if the sum of lengths of two opposing sides is equal to the sum of lengths of the other two opposing sides, i.e. $|AB| + |CD| = |AD| + |BC|$.




**Proof**. (1) First, we prove that the condition is necessary: assume there is a circle inscribed into the quadrilateral, and it has common points $M, N, P, Q$ with sides $AB$, $BC$, $CD$, and $AD$, respectively. Then, using Theorem 29, we can immediately see that $|AM| = |AQ|$, $|BM| = |BN|$, $|CN| = |CP|$, and $|DP| = |DQ|$ and the condition is satisfied.

(2) Now let's prove that if $|AB| + |CD| = |AD| + |BC|$ then a circle can be inscribed into a quadrilateral $ABCD$. First, consider the case $|AB| < |BC|$, and then $|CD| > |AD|$. On side $BC$ mark point $M$ such that $|BM| = |BA|$ and, similarly, on side $CD$ mark point $F$ such that $|DF| = |DA|$. Note that by Theorem 30 also $|CF| = |CM|$, and therefore $\triangle\, ADF$, $\triangle\, FCM$, and $\triangle\, MBA$ are isosceles. Consider perpendicular bisectors to the sides of $\triangle\, AFM$: they must intersect at the same point $O$. At the same time, they are bisectors of angles $\angle ADC$, $\angle DCB$, and $\angle CBA$. Therefore, the intersection point $O$ is equidistant from the four sides of quadrilateral $ABCD$ and a circle with center $O$ can be inscribed into it. The proof of the case $|AB| > |BC|$ is completely analogous, and the case $|AB| = |BC|$ is left to you as an exercise. □




The property that the sides of a triangle or a quadrilateral are equidistant from the center of the inscribed circle allows one to calculate the area:

**Theorem 31.** The area of a polygon with inscribed circle of radius $r$ and perimeter $P$ is $S = \frac{1}{2}Pr$.

**Proof**. Left as a homework exercise. □

## Homework problems

**Note that you may use all results that are presented in the previous sections**. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. Show that
   a. a parallelogram can be circumscribed with a circle <u>if and only if</u> it is a rectangle;
   b. a trapezoid $ABCD$ ($AD \parallel BC$) can be circumscribed <u>if and only if</u> it is isosceles (its sides are equal $|AB| = |CD|$, or, equivalently, the angles at the same base are equal: $\angle DAB = \angle ADC$ or $\angle ABC = \angle DCB$).
2. Finish the proof of Theorem 30 for the case $|AB| = |BC|$.
3. Prove Theorem 31.
4. In what kind of parallelogram can a circle be inscribed?
5. Can you inscribe a circle into a quadrilateral with sides (in order)
   a. 2cm, 2cm, 3cm, 3cm?
   b. 5cm, 3cm, 1cm, 3cm?
   c. 2cm, 5cm, 3cm, 4cm?
6. What is the area of a trapezoid with sides 5cm and 7cm and the radius of inscribed circle 2cm?
7. What is the area of a quadrilateral with two adjacent sides $a$ and $b$, angle between them $\alpha$, and the radius of inscribed circle $r$?
8. Can one circumscribe a circle around a quadrilateral $ABCD$ if the ratios of angles $\angle A$: $\angle B$ : $\angle C$ $\angle$: $D$ are
   a. 2 : 3 : 4 : 3?
   b. 7 : 2 : 4 : 5?