January 19, 2025 Math 8

Handout 17. Euclidean Geometry 6: Circles.

Circles.

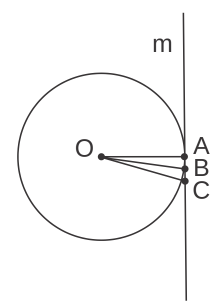
Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $|OP| = r$.

Traditionally, circles are denoted by Greek letters: $\lambda, \omega, \ldots$. Given a circle λ with center O,

- A radius is any line segment from O to a point $A \in \lambda$,
- A chord is any line segment between distinct points $A, B \in \lambda$,
- A diameter is a chord that passes through the center, O,
- A line is tangent if it has exactly one common point with the circle and is said to be the tangent through that point,
- Two circles are tangent if they have exactly one common point.

Theorem 21. Let A be a point on circle λ centered at O, and m a line through A. Then m is tangent to λ if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to λ at A.

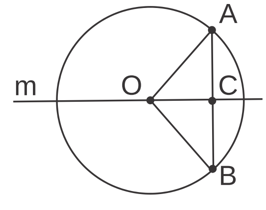
Proof. First, we prove $(m\ is\ tangent\ to\ \lambda) \Rightarrow (m \perp \overline{OA})$ using the method of contradiction. Suppose m is tangent to λ at A but not perpendicular to $\overline{OA}$. Let $\overline{OB}$ be the perpendicular to m through O, with $B \in m$. Construct point $C \in m$ such that $|BA| = |BC|$; then we have that $\triangle OBA \cong \triangle OBC$ by SAS, using $|OB| = |OB|$, $\angle OBA = \angle OBC = 90°$, and $|BA| = |BC|$. Therefore $|OC| = |OA|$ and hence C is on λ. But this means that m intersects λ at two points, which is a contradiction.



Now, we prove $(m \perp \overline{OA}) \Rightarrow (m\ is\ tangent\ to\ \lambda)$. Suppose m passes through $A \in \lambda$ such that $m \perp \overline{OA}$. If m also passed through B on λ, then $\triangle AOB$ would be an isosceles triangle since $\overline{AO}, \overline{BO}$ are radii of λ. Therefore, $\angle ABO = \angle BAO = 90°$, i.e. $\triangle AOB$ is a triangle with two right angles, which is a contradiction. □

Notice that, given point O and line m, the perpendicular $\overline{OA}$ from O to m (with $A \in m$) is the shortest distance from O to m, therefore the locus of points of distance exactly $|OA|$ from O should line entirely on one side of m. This is essentially the idea of the above proof. □

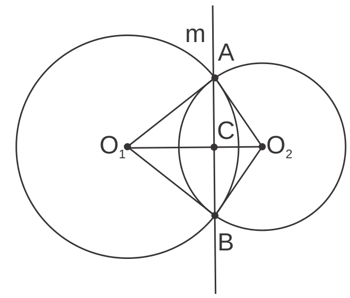
Theorem 22. Let AB be a chord of a circle λ with center O. Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if $C \in \overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.



Proof. Let m be the perpendicular bisector of $\overline{AB}$. The center O of λ is equidistant from A, B by the definition of a circle, therefore O must be on m. Let m intersect $\overline{AB}$ at D. We then have that D is the midpoint of $\overline{AB}$ and also the foot of the perpendicular from O to $\overline{AB}$.

Then, if C bisects $\overline{AB}$, C lies on the perpendicular bisector m of $\overline{AB}$, which passes through O, thus $\overline{OC} \perp \overline{AB}$. Lastly if $\overline{OC} \perp \overline{AB}$, then because there is only one perpendicular to $\overline{AB}$ through O, we must have $C = D$ and hence C is the midpoint of $\overline{AB}$. □

Theorem 23. Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.



Proof. Let l be the perpendicular bisector of $\overline{AB}$. By the previous theorem, l contains both centers: $O_1 \in l, O_2 \in l$. Thus, $l = \overline{O_1O_2}$, so $\overline{O_1O_2}$ is the perpendicular bisector of $\overline{AB}$. Hence, $\overline{AB} \perp \overline{O_1O_2}$. □

Theorem 24. [Relative positions of lines and circles] Let λ be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular $\overline{OP}$ from O to l. Then:

- If $d > r$, then λ and l do not intersect.
- If $d = r$, then λ intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to λ at P.
- If $d < r$, then λ intersects l at two distinct points.

Proof. First two parts easily follow from the fact that a perpendicular is the (shortest) distance from a point to a line. In the last part, it is easy to show that λ can not intersect l at more than 2 points. Proving that it does intersect l precisely at two points is very hard and requires deep results about real numbers. This proof will not be given here. □

Note that it follows from the definition and Theorem 21 that a tangent line is perpendicular to the radius $\overline{OP}$ at point of tangency. Converse is also true.

Theorem 25. Let ω_1, ω_2 be circles that are both tangent to the same line m at point A. Then these two circles have only one common point, A. Such circles are called tangent.

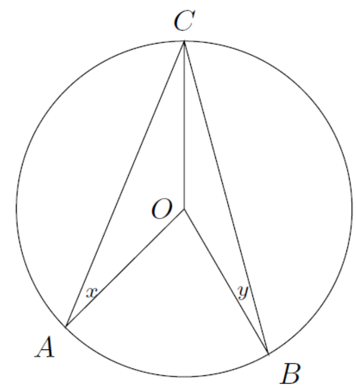
Proof. By Theorem 21, radiuses $\overline{O_1A}$ and $\overline{O_2A}$ are both perpendicular to m at A; since there can only be one perpendicular line to m at a given point, it means that O_1, O_2, and A are on the same line, and that $m \perp \overline{O_1O_2}$, i.e, m is perpendicular to $\overline{O_1O_2}$ at A.

Now, suppose that ω_1, ω_2 intersect at point $B \neq A$. Then by the previous theorem, $\overline{AB} \perp \overline{O_1O_2}$, therefore both $\overline{AB}$ and m are perpendicular to $\overline{O_1O_2}$ through A. We must therefore have that B is on m, but m is tangent to ω_1 through A, thus has only one intersection with ω_1, which is a contradiction. □

Arcs and angles.

Consider a circle λ with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle inside this angle is called the arc subtended by $\angle AOB$.

Theorem 26. Let A, B, C be on a circle λ with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ B is said to be inscribed in λ.



Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the right.

Case 1. Draw in segment $\overline{OC}$. Denote $m\angle A = x$, $m\angle B = y$. Since $\triangle AOC$ is isosceles, $m\angle ACO = x$; similarly $m\angle BCO = y$, so $m\angle ACB = x + y$, and $m\angle AOC = 180° - 2x$, $m\angle BOC = 180° - 2y$. Therefore, $m\angle AOC + m\angle BOC = 360° - 2(x + y)$. This implies $m\angle AOB = 2(x + y)$. □

As a result of Theorem 26, we get that any triangle $\triangle ABC$ on λ where $\overline{AB}$ is a diameter must be a right triangle, since the angle $\angle ACB$ has half the measure of angle $\angle AOB$, which is 180°.

The idea captured by the concept of an arc and Theorem 26 is that there is a fundamental relationship between angles and arcs of circles, and that the angle 360° can be thought of as a full circle around a point.

Homework problems

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. Prove that given a segment $\overline{AB}$, there is a unique circle with diameter $\overline{AB}$.
2. Given lines $\overline{AB} \perp \overline{CD}$ such that $\overline{AD}, \overline{BC}$ intersect at E and $|AE| = |ED|$, prove that $|BE| = |EC|$.
3. Prove that if a diameter of circle λ is a radius of circle ω, then λ, ω are tangent.
4. Complete the proof of Theorem 26 by proving the cases where O is not inside the angle $\angle ACB$. [Hint: for one of the cases, you may need to write $\angle ACB$ as the difference of two angles.]
5. Prove the converse of Theorem 26: namely, if λ is a circle centered at O and A, B, are on λ, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ. [Hint: we need to prove that $|OC| = |OA|$; consider using a proof by contradiction]
6. Let A, B be on circle λ centered at O and m the tangent to λ at A. Let C be on m such that C is on the same side of $\overline{OA}$ as B. Prove that $m\angle BAC = \frac{1}{2}m\angle BOC$. [Hint: extend $\overline{OA}$ to intersect λ at point D so that $\overline{AD}$ is a diameter of λ. What arc does $\angle DAB$ subtend?]
7. Prove that, given two distinct points A, B on circle λ which are on the same side of diameter $\overline{CD}$ of λ, $CB \neq CA$.
8. Let $\overline{AB}, \overline{CD}$ both have midpoint E and let F, G be points such that $BECF$ and $AEDG$ are parallelograms. Prove that E is the midpoint of FG.