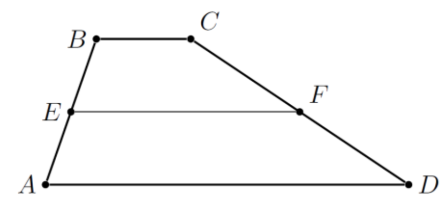
Handout 16. Euclidean Geometry 4: Quadrilaterals: Trapezoid. Concurrence.

Special quadrilaterals: Trapezoid.

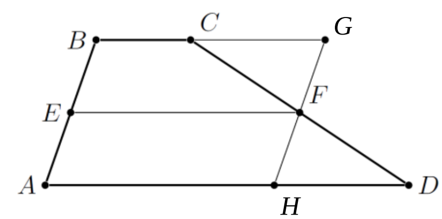
Definition. A quadrilateral is called a trapezoid, if one pair of opposite sides are parallel (these sides are called bases), while the other pair maybe not.

If the other two sides are also parallel, then the quadrilateral is a parallelogram, so all theorems that apply to a trapezoid will also apply to a parallelogram, although some may become trivial. The most interesting property of a trapezoid is its midline.

Definition. The midline of a trapezoid $ABCD$ ($AD \parallel BC$) is the segment (EF) connecting the midpoints (E and F) of its sides (AB and CD).



Theorem 18. [Trapezoid midline] Let $ABCD$ be a trapezoid, with bases AD and BC, and let E, F be midpoints of sides AB, CD respectively. Then $EF \parallel AB$, and $|EF| = (|AD| + |BC|)/2$.



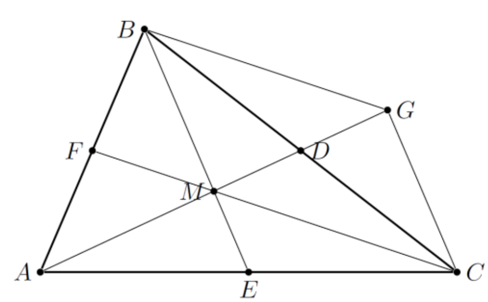
Proof. Draw through point F a line parallel to AB, as shown in the figure. The intersection points of this line with the base AD (H) and continuation of the base BC (G) gives a parallelogram, $ABGH$, in which points E, F are the midpoints of the opposite sides. Hence, $EF \parallel AD \parallel BC$ and $|EF| = |AH| = |BG|$.

Exercise. (homework) Complete the proof, showing that $|EF| = (|AD| + |BC|)/2$. □

Of course, the above theorem is automatically fulfilled for a parallelogram, and the midline of a parallelogram is congruent to the sides it is parallel to.

Intersection point of medians.

Theorem 19. [Intersection point of medians in a triangle] Let ABC be a triangle and AD, BE, and CF be its medians. Then AD, BE, and CF intersect at a single point M (are concurrent) and each is divided by point M in 2: 1 ratio counting from their respective vertices: $|AM|:|MD| = |BM|:|ME| = |CM|:|MF| = 2:1$.



Proof. First, let's prove that if BE and CF are medians intersecting at point M, and AD intersects them at the same point, then AD is also a median.

To prove this, continue line AD beyond point D and mark point G such that $|GM| = |AM|$. Note that,

1. M is the midpoint of AG, and E is the midpoint of AC. Therefore, ME is a midline of $\triangle AGC$ and $ME \parallel GC$.
2. Similarly, MF is a midline of $\triangle AGB$ and $MF \parallel GB$.

From the above, it follows that $BMGC$ is a parallelogram, and its diagonals, BC and MD, bisect each other. Hence, D is the midpoint of BC and AD is a median.

Exercise. (homework) Complete the proof, showing that $|AM| = 2|MD|$, $|BM| = 2|ME|$, $|CM| = 2|MF|$ □

By now, we have proven that the following lines in any triangle are concurrent (intersect at the same point):

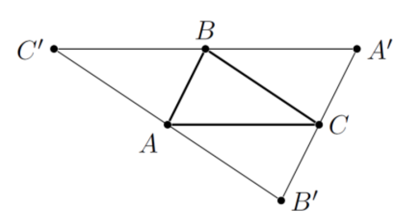
- the three angle bisectors intersect at the same point (incenter), which is equidistant from the three sides of the triangle;
- the three perpendicular side bisectors intersect at the same point (circumcenter), which is equidistant from the three vertices (corners) of the triangle;
- the three altitudes intersect at the same point, which is called the orthocenter, and may be inside or outside the triangle;
- and the three medians intersect at the same point, which is called the centroid, and are divided by it 2: 1 counting from the triangle vertices.

The **centroid of a triangle** (intersection point of the medians) has a remarkable property: it **is a center of mass of a uniform triangle**. You can check this by cutting out a triangle from a sheet of cardboard or other uniform material and balancing it on the tip of a needle. The same point will also be the center of mass if you place **three equal masses** at each vertex.

Homework problems

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

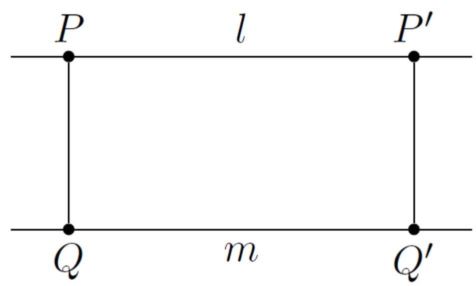
1. Finish the proof of Theorem 18: show that the length of the midline of a trapezoid is the arithmetic mean of its bases, $|EF| = (|AD| + |BC|)/2$.
2. Finish the proof of Theorem 19: show that the intersection point splits medians in 2 : 1 ratio, counting from the vertex.
3. Review the proof that the three altitudes of a triangle intersect at a single point. Given a triangle $\triangle ABC$, draw through each vertex line parallel to the opposite side. Denote the intersection points of these three lines by A', B', C', as shown in the figure.
    a. Prove that $|A'B| = |AC|$ (hint: use parallelograms!)
    b. Show that B is the midpoint of $A'C'$, and similarly for other two vertices.



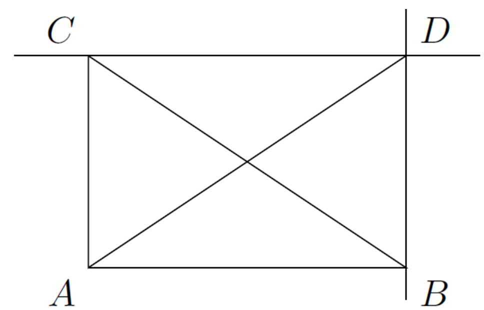
c. Show that the altitudes of $\triangle ABC$ are exactly the perpendicular bisectors of the sides of $\triangle A'B'C'$.

d. Prove that the three altitudes of $\triangle ABC$ intersect at a single point.

4. (Distance between parallel lines) Let l, m be two parallel lines. Let $P \in l$, $Q \in m$ be two points such that $PQ \perp l$ (by Theorem 6, this implies that $PQ \perp m$). Show that then, for any other segment $P'Q'$, with $P' \in l$, $Q' \in m$ and $P'Q' \perp l$, we have $|P'Q'| = |PQ|$. (This common distance is called the distance between l, m).



5. Let $\triangle ABC$ be a right triangle ($m\angle A = 90°$) and let D be the intersection of the line parallel to $\overline{AB}$ through C with the line parallel to $\overline{AC}$ through B.
   a. Prove $\triangle ABC \cong \triangle DCB$
   b. Prove $\triangle ABC \cong \triangle BDA$
   c. Prove that AD is a median of $\triangle ABC$.



6. Let $\triangle ABC$ be a right triangle ($m\angle A = 90°$) and let A' be the midpoint of $\overline{BC}$. Prove that $|AA'| = \frac{1}{2}|BC|$.