January 5, 2025 Math 8

Handout 15. Euclidean Geometry 4: Quadrilaterals. Midline of a triangle.

Special quadrilaterals.

In general, a figure (polygon) with four sides (and four enclosed angles) is called a quadrilateral; by convention, their vertices are labeled in order going around the perimeter (so, for example, in quadrilateral $ABCD$, vertex A is opposite vertex C). In case it is unclear, we use 'opposite' to refer to pieces of the quadrilateral that are on opposite sides, so side $\overline{AB}$ is opposite side $\overline{CD}$, vertex A is opposite vertex C, angle $\angle A$ is opposite angle $\angle C$, etc..

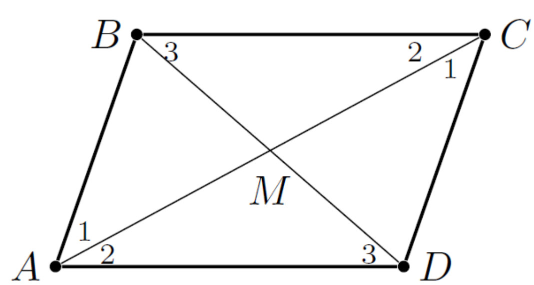
Definition. A polygon with 4 vertices is called quadrilateral.

Among all quadrilaterals, there are some that have special properties. In this section, we discuss three such types.

Definition. A quadrilateral is called

- a parallelogram, if both pairs of opposite sides are parallel
- a rhombus, if all four sides have the same length
- a trapezoid, if one pair of opposite sides are parallel (these sides are called bases) and the other pair is not.

These quadrilaterals have several useful properties.



Theorem 14. Let $ABCD$ be a parallelogram. Then:

- $AB = DC, AD = BC$
- $m\angle A = m\angle C, m\angle B = m\angle D$
- The intersection point M of diagonals AC and BD bisects each of them.

Proof. Consider triangles $\triangle ABC$ and $\triangle CDA$ (pay attention to the order of vertices!). By Axiom 4 (alternate interior angles), angles $\angle CAB$ and $\angle ACD$ are equal (they are marked by 1 in the figure); similarly, angles $\angle BCA$ and $\angle DAC$ are equal (they are marked by 2 in the figure). Therefore, by ASA, $\triangle ABC \cong \triangle CDA$. Thus, $AB = DC$, $AD = BC$, and $m\angle B = m\angle D$. Similarly, one can prove that $m\angle A = m\angle C$.

Now let us consider triangles $\triangle AMD$ and $\triangle CMB$. In these triangles, angles labeled 2 are congruent (as discussed above), and by Axiom 4, angles marked by 3 are also congruent; finally, $AD = BC$ by previous part. Therefore, $\triangle AMD \cong \triangle CMB$ by ASA, so $AM = MC, BM = MD$. □

There are several ways you can recognize a parallelogram; in fact, conclusions in Theorem 14 are not only necessary, but also sufficient for a quadrilateral to be a paralellogram. Let's remember them as a single theorem:

Theorem 15. A quadrilateral $ABCD$ is a parallelogram **if and only if** one of the following holds:

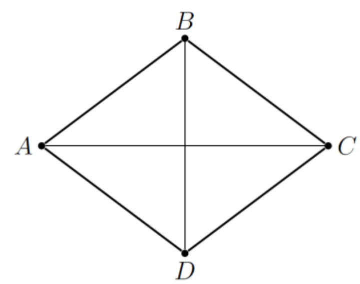
- its opposite sides are equal ($AB = CD$ and $AD = BC$), OR
- two opposite sides are equal and parallel ($AB = CD$ and $AB \parallel CD$), OR
- its diagonals bisect each other ($AM = CM$ and $BM = DM$, where $M = AC \cap BD$), OR
- its opposing angles are equal ($\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$).

Proof. Left as a homework exercise. □

Definition. A quadrilateral all of whose sides are congruent is called rhombus.

Theorem 16. Let $ABCD$ be a rhombus. Then it is a parallelogram; in particular, the intersection point of diagonals is the midpoint for each of them. Moreover, the diagonals are perpendicular.

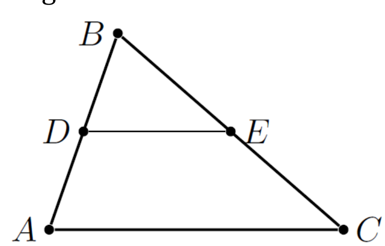
Proof. Since the opposite sides of a rhombus are equal, it follows from Theorem 15 that the rhombus is a parallelogram, and thus the diagonals bisect each other. Let M be the intersection point of the diagonals; since triangle $\triangle ABC$ is isosceles, and BM is a median, by Theorem 12, it is also the altitude. □



Midline of a triangle.

Properties of parallelograms are very useful for proving theorems, among which are important properties of a triangle midline.

Definition. The segment connecting the midpoints of two sides of a triangle $\triangle ABC$ is called a midline of $\triangle ABC$.

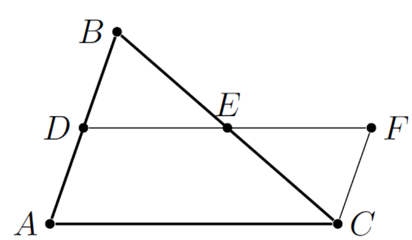


Theorem 15. If DE is the midline of $\triangle ABC$, then $DE = \frac{1}{2}AC$, and $DE \parallel AC$.

Proof. Continue line DE and mark on it point F such that $DE = EF$.

1. $\triangle DEB \cong \triangle FEC$ by SAS: $DE = EF$, $BE = EC$, $\angle BED \cong \angle CEF$.

2. $ADFC$ is a parallelogram: First, we can see that since $\triangle DEB \cong \triangle FEC$, then $\angle BDE \cong \angle CFE$, and since they are alternate interior angles, $AD \parallel FC$. Also, from the same congruency, $FC = BD$, but $BD = AD$ since D is a midpoint. Then, $FC = DA$. So we have $FC = DA$ and $FC \parallel DA$, and therefore $ADFC$ is a parallelogram.



3. That gives us the second part of the theorem: $DE \parallel AC$. Also, since $ADFC$ is a parallelogram, $AC = DF = 2 \cdot DE$, and from here we get $DE = \frac{1}{2}AC$. □

Alternatively, one can prove that if a line parallel to one side of the triangle crosses another side in the middle, then it is a midline and will cross the third side also in the middle.

Homework problems

Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. Prove that in a parallelogram, sum of two adjacent angles is equal to 180°:

$$m\angle A + m\angle B = m\angle B + m\angle C = \cdots = 180°.$$

2. [We may have done some in class - you still need to write the proofs neatly] Prove Theorem 15, that a quadrilateral is a parallelogram if:
    a. it has two pairs of equal sides;
    b. if two of its sides are equal and parallel;
    c. if its diagonals bisect each other;
    d. if its opposite angles are equal.

Any of the above statements can be used as the definition of a parallelogram.

3. (Rectangle) A quadrilateral is called rectangle if all angles have measure 90°.
    a. Prove that opposite sides of a rectangle are congruent.
    b. Prove that the diagonals of a rectangle are congruent.
    c. Prove that a rectangle is a parallelogram.
    d. Prove that conversely, if $ABCD$ is a parallelogram such that $AC = BD$, then it is a rectangle.
4. Prove that in any triangle, the three perpendicular side bisectors intersect at a single point (compare with the similar fact about angle bisectors – Problem 3 from Handout 16).
5. Show that in any triangle, its three midlines divide the original triangle into four triangles, all congruent to each other.
6. Prove that in any triangle, its altitudes intersect at the same point. [Hint: consider a triangle and its three midlines from the previous problem; draw the perpendicular side bisectors to each side of the big triangle. Are they altitudes in some other triangle?]