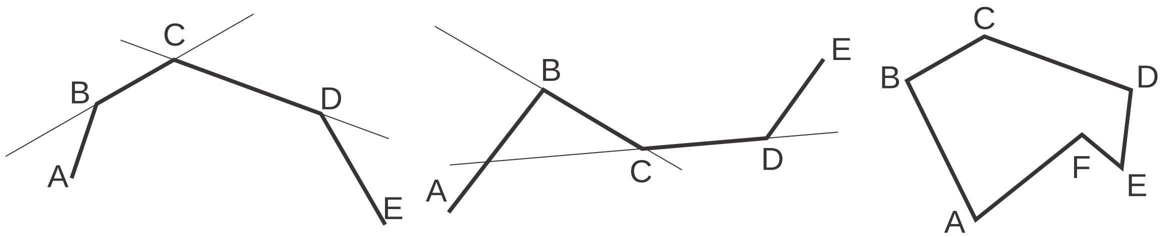
December 15, 2024 Math 8

# Handout 14. Euclidean geometry 3: Triangle inequalities.

## Broken lines and polygons.

After we have introduced elementary objects, including undefined ones, and axioms and common notions that describe their properties, we can proceed with building up the Euclidean geometry. We will do so by introducing more complex objects and formulating and proving theorems which specify more complex properties of objects and relations among them.

**Definition**. A set of connected straight segments not all lying on a straight line, such that each two consecutive segments share an end point, form a **broken line**. A broken line is **convex** if it lies on one side of all the straight lines containing each of its segments. Otherwise, the line is **concave**. A broken line whose endpoints coincide is called **closed**.




**Definition**. A set of points on the plane bounded by a non-intersecting closed broken line is called **polygon**. In other words, a polygon is the figure formed by a non-intersecting broken line and the part of the plane bounded by it. The straight segments constituting the broken line are called **sides** of the polygon and their endpoints are **vertices**. The angles formed by the adjacent sides sharing a vertex are called (**interior**) **angles** of the polygon. A polygon is convex if it is formed by a convex closed broken line, otherwise it is concave. The broken line itself is called the **boundary** of the polygon, and the total length of its segments the **perimeter**. Polygons with small number of vertices have special names. The smallest possible number of vertices is 3, such polygons are called triangles; polygons with 4 vertices are quadrilaterals, with 5 pentagons, with 6 hexagons, and so on.

## Triangles. Isosceles triangles. Equilateral triangle.

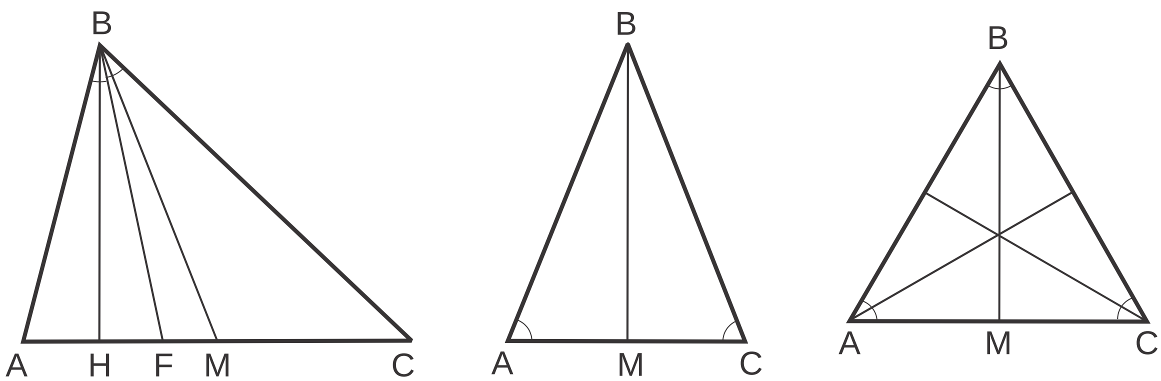
The first and one of the most important geometrical figures we consider is a triangle. Any polygon can be represented as a combination of triangles.

**Exercise 1**. What is the sum of the angles of a convex quadrilateral? Pentagon? Hexagon? $n$-gon?

**Definition**. A triangle is a polygon with three sides (and three vertices, and three angles). Alternatively, triangle is a set of points on the plane bounded by the three segments connecting three given points.

**Lines in a triangle**. In any triangle, there are three special lines from each vertex.

In a $\triangle ABC$, the segment $(BH)$ connecting a vertex with the opposite side (base) and perpendicular to that side is called an **altitude** (it exists and is unique by Theorem about the existence of the perpendicular). The segment $(BM)$ connecting a vertex with the midpoint of the opposite side (base) is called a **median**. The segment $(BF)$ connecting a vertex with the opposite side (base) and dividing the angle at the vertex in two equal halves, $\angle ABF \cong \angle FBC$, is called a **bisector**. For general triangle, all three lines are different. However, as we will see below, in some triangles they coincide.




**Definition**. A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base. While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

**Theorem 10**. In an isosceles triangle, the bisector of the angle at the apex (vertex opposite the base) is at the same time the median and the altitude.

**Proof**. Consider an isosceles triangle $\triangle ABC$ with a median $BM$ from apex $B$. We observe that $AB \cong CB$ (by definition of isosceles triangle), $AM \cong CM$ (by definition of midpoint), and side $BM$ is shared by both triangles. It then follows from the SSS congruence theorem that $\triangle ABC \cong \triangle DEF$. Then, by SAS axiom, $m\angle ABM = m\angle CBM$, so $BM$ is the bisector of the angle $\angle ABC$. It also follows that $m\angle AMB = m\angle CMB$. On the other hand, $m\angle AMB + m\angle CMB = 180°$. It then follows that $m\angle AMB = m\angle CMB = 90°$, so $BM$ is the altitude. ☐

**Theorem 11** (base angles equal). If $\triangle ABC$ is isosceles with base $AC$, then $m\angle A = m\angle C$ (i.e., $\angle BAC \cong \angle BCA$). Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles with base $AC$.

The direct and converse theorem provide the necessary and sufficient conditions for a triangle to be isosceles, which can be formulated as,

$$(\triangle ABC \textit{ is isosceles with base } AC) \Leftrightarrow (m\angle A = m\angle C)$$

**Proof**. Left as a homework exercise. We need to prove both necessary and sufficient condition. ☐

**Definition (axial symmetry)**. If two points, $A$ and $C$, are on the opposite sides of a line $a = \overleftrightarrow{BH}$ which is perpendicular to $AC$ and are the same distance away from the foot of the perpendicular, $H$, i.e. $AH \cong BH$, then points $A$ and $C$ are called **symmetric** with respect to the line $a$.

Two figures (or two parts of a figure) are symmetric with respect to line $a$ if for each point of one figure (or one part of a figure) there is a symmetric point in the other figure (or the other part of the figure) and *vice versa*. The line $a$ is called the **axis of symmetry**.

## Triangle inequalities.

Now we can proceed with proving some important properties of triangles which underlie great number of practical applications of Euclidean geometry. In this section, we use previous results about triangles to prove two important inequalities which hold for any triangle.

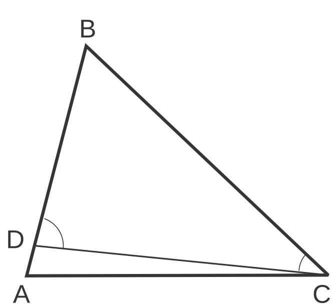
We already know that if two sides of a triangle are equal, then the angles opposite to these sides are also equal (this is a property of an isosceles triangle we proved). The next theorem extends this result: in a triangle, if one angle is bigger than another, the side opposite the bigger angle must also be longer than the one opposite the smaller angle.

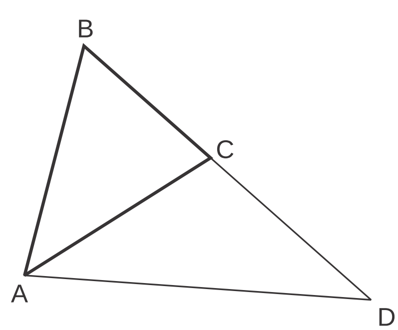
**Theorem 12.** In $\triangle ABC$ , if $m\angle A > m\angle C$, then we must have $|BC| > |AB|$. Conversely, if $|BC| > |AB|$ then $m\angle A > m\angle C$. Or, using logic notations,

$$\forall \triangle ABC, (m\angle A > m\angle C) \Leftrightarrow (|BC| > |AB|)$$

**Proof.** First, we prove $(m\angle A > m\angle C) \Rightarrow (|BC| > |AB|)$, i.e., if $m\angle A > m\angle C$, then $|BC| > |AB|$ using proof by contradiction. Assume that $(m\angle A > m\angle C) \wedge (|BC| \leq |AB|)$. If $|BC| = |AB|$, then $\triangle ABC$ is isosceles with the base $AC$ and, according to theorem 11, $m\angle A = m\angle C$, which contradicts $m\angle A > m\angle C$. Assume now $|BC| < |AB|$. Find the point $D$ on $AB$ such that $|BD| = |BC|$, and draw the segment $CD$. $\triangle BCD$ is isosceles with the apex $B$ and, therefore, $m\angle BDC = m\angle BCD$. On the other hand, $m\angle BCD < m\angle C$ (this easily follows from Angle Measurement Axiom) and $m\angle A < m\angle BDC$ because $\angle BDC$ is an external angle of $\triangle ACD$ and therefore is larger than any internal angle of that triangle. We thus obtain, $m\angle A < m\angle BDC = m\angle BCD < m\angle C$, which contradicts $m\angle A > m\angle C$.

**Exercise 2.** (homework). Prove $(|BC| > |AB|) \Rightarrow (m\angle A > m\angle C)$ □




This leads us to a proof of one of the most important theorems in Euclidean geometry, which also has counterparts in linear algebra and vector analysis (Cauchy–Schwarz inequality), and other branches of mathematics.

**Theorem 13 (the triangle inequality).** In a triangle, a side is less than the sum of the two other sides,

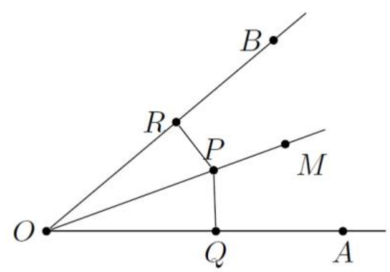
$$\forall \triangle ABC, |AB| < |BC| + |CA|$$

**Proof**. Extend the line $BC$ past $C$ to the point $D$ so that $|AC| = |CD|$ and join the points $A$ and $D$ with a line so as to form the triangle $\triangle ABD$. Observe that $\triangle ACD$ is isosceles with apex at $C$; hence $m\angle CAD = m\angle CDA$. It immediately follows that $m\angle BAD = m\angle BAC + m\angle CAD > m\angle CDA$. Then, by Theorem 12, this implies $|BD| > |AB|$. Our result now follows from $|BD| = |BC| + |CD|$ (Axiom 2) and $|AC| = |CD|$ (by construction). □

## Homework problems

**Note that you may use all results that are presented in the previous sections**. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms.

1. (Slant lines and perpendiculars) Let $P$ be a point not on line $l$, and let $Q \in l$ be such that $PQ \perp l$. Prove that then, for any other point $R$ on line $l$, we have $PR > PQ$, i.e. the perpendicular is the shortest distance from a point to a line. Note: you cannot use the Pythagorean theorem for this, as we haven't yet proved it! Instead, use Theorem 12.
2. Review the proof of Theorem about the sum of angles of a triangle and solve Exercise 1 about the sum of the angles of a convex polygon.
3. (Angle bisector). Define a distance from a point $P$ to line $l$ as the length of the perpendicular from P to $l$ (compare with the previous problem). Let $\overrightarrow{OM}$ be the angle bisector of $\angle AOB$, i. e. $\angle AOM \cong \angle MOB$.




   a. Let $P$ be any point on $\overrightarrow{OM}$, and $PQ$, $PR$ – perpendiculars from $P$ to sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively. Use ASA axiom to prove that triangles $\triangle OPR$ and $\triangle OPQ$ are congruent and deduce from this that distances from $P$ to $\overrightarrow{OA}$, $\overrightarrow{OB}$ are equal.
   b. Prove that conversely, if $P$ is a point inside angle $\angle AOB$, and distances from $P$ to the two sides of the angle are equal, then $P$ must lie on the angle bisector, $\overrightarrow{OM}$.

These two statements show that the locus of points equidistant from the two sides of an angle is the angle bisector.

4. Prove that in any triangle, the three angle bisectors intersect at a single point (compare with the similar fact about perpendicular bisectors).