December 8, 2024 Math 8

Handout 13. Euclidean geometry 2: Axioms and Theorems.

Euclidean Geometry: Axioms.

After we introduced elementary objects, including undefined ones, we need to have statements (axioms) that describe their properties. Of course, the lack of definition for undefined objects makes such properties impossible to prove. The goal here is to state the minimal number of such properties that we take for granted, called axioms, just enough to be able to prove or derive harder and more complicated statements, theorems.

As a basis for further logical deductions, Euclid proposed to use five "common notions" such as "things equal to the same thing are equal," and five unprovable but intuitive principles known variously as postulates or axioms.

Common notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

Axiom 1. Given two points, there is a straight line that joins them.

Axiom 2. A straight line segment can be prolonged indefinitely.

Axiom 3. A circle can be constructed when a point for its center and a distance for its radius are given.

Axiom 4. All right angles are equal.

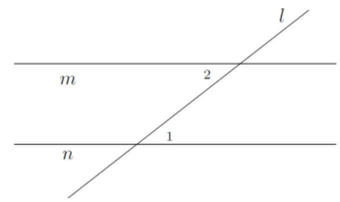
Axiom 5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which the angles are less than the two right angles.

There are many other, equivalent ways to state the same axioms. Hilbert refined axioms (1) and (5), and stated in modern terms these axioms are as follows:

Axiom 1. For any two different points, (a) there exists a line containing these two points, and (b) this line is unique. That is, for any two distinct points A, B, there is exactly one, one and only one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.

Axiom 5. For any line l and point p not on l, (a) there exists a line through p not meeting l, and (b) this line is unique.

Here is yet another way to state the same Axiom 5, which we will use:



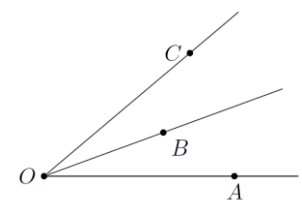
Axiom 5. Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in the figure (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.

This fifth axiom is also known as the "parallel postulate," since it provides a basis for the uniqueness of parallel lines. It attracted great interest because unlike Euclid's other four postulates, it never seemed entirely self-evident and there were many efforts to prove it through the centuries. This quest has ended in the 19th century when Nikolay Lobachevsky followed by János Bolyai and Carl Friedrich Gauss discovered that altering the parallel postulate resulted in perfectly consistent non-Euclidean geometries thus breaking the uniqueness of Euclidean geometry.

All five axioms provide the basis for numerous provable statements, or theorems, on which Euclid built his geometry. We will supplement these axioms with another two.

Axiom 6. If distinct points A, B, C, are on the same line, exactly one is between the other two; if point B is between A and C, then $AC = AB + BC$.

Axiom 7. If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to 180 degrees.



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

Euclidean Geometry: First Theorems.

Now we can proceed with proving some results based on the axioms above.

Theorem 1. If distinct lines l, m intersect, then they intersect at exactly one point.

Proof. Proof by contradiction: Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more than one point) must be false. □

Theorem 2. Given a line l and point P not on l, there exists a unique line m through P which is parallel to l.

Proof. This, of course, is just the Hilbert formulation of the Axiom 5. As an axiom, it cannot be proven. However, if we accept another formulation of Axiom 5, then this statement becomes a theorem which needs to be proven. Here, we use the last formulation of the axiom 5 given above and prove this one as a theorem. We can also accept this formulation as an axiom and then prove the other one. Doing so will establish the equivalence of the two formulations of Axiom 5.

We have to prove two things: the existence of a parallel line through the given point not on the given line, and its uniqueness. Below we provide a sketch of the proof – please fill in the details and draw a diagram at home!

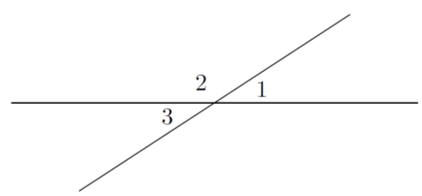
Existence: Let m be any line that goes through P and intersect l at point O. Let A be a point on the line l. Then we can measure the angle $\angle POA$. Now, let PB be such that $m\angle PBO = m\angle POA$ and B is on the other side of m than A. In this case, by Axiom 5, $\overleftrightarrow{PB} \parallel l$.

Uniqueness: Imagine that there are two lines m, n that are parallel to l and go through P. Take a line k that goes through P and intersects l at point O. Let A be a point on line l distinct from O, and B, C – points on lines m and n respectively on the other side of line k than A. Since both m, n are parallel to l, we can see that $m\angle AOP = m\angle BPO = m\angle CPO$ – but that would mean that lines BP and CP are the same, in contradiction to our assumption that there are two such lines. □

Theorem 3. If $l \parallel m$ and $m \parallel n$, then $l \parallel n$.

Proof. Assume that l and n are not parallel and intersect at point P. But then it appears that there are two lines that are parallel to m are go through point P – contradiction with Theorem 2 (or Axiom 5 in Hilbert's formulation). □

Theorem 4. Let A be the intersection point of lines l, m, and let angles 1,3 be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.



Proof. Let angle 2 be as shown in the figure to the left. Then, by Axiom 7, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 5. Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to 90°. Then the three other angles are also equal to 90° (In this case, we say that lines l, m are perpendicular and write $l \perp m$.)

Proof. Left as a homework exercise. □

Theorem 6. Let l_1, l_2 be perpendicular to m. Then $l_1 \perp l_2$. Conversely, if $l_1 \perp m$ and $l_2 \perp l_1$, then $l_2 \perp m$.

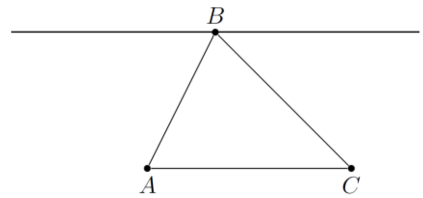
Proof. Left as a homework exercise. □

Theorem 7. Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.

Proof. Left as a homework exercise. □

Triangles.

Theorem 8. Given any three points A, B, C, which are not on the same line, and line segments AB, BC, and CA, we have $m\angle ABC + m\angle BCA + m\angle CAB = 180°$. (Such a figure of three points and their respective line segments is called a triangle, written $\triangle ABC$. The three respective angles are called the triangle's interior angles.)



Proof. The proof is based on the figure and use of Alternate Interior Angles axiom. Details are left to you as a homework.

Congruence.

It will be helpful, in general, to have a way of comparing geometric objects to tell whether they are the same. We will build up such a notion and call it congruence of objects. To begin, we define congruence of angles and congruence of line segments (note that an angle cannot be congruent to a line segment; the objects have to be the same type).

- If two angles $\angle ABC$ and $\angle DEF$ have equal measure, then they are congruent angles, written $\angle ABC \cong \angle DEF$.
- If the distance between points A, B is the same as the distance between points C, D, then the line segments $\overline{AB}$ and $\overline{CD}$ are congruent line segments, written $\overline{AB} \cong \overline{CD}$.
- If two triangles $\triangle ABC$, $\triangle DEF$ have respective sides and angles congruent, then they are congruent triangles, written $\triangle ABC \cong \triangle DEF$. In particular, this means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{CA} \cong \overline{FD}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$.

Note that congruence of triangles is sensitive to which vertices on one triangle correspond to which vertices on the other. Thus, $\triangle ABC \cong \triangle DEF \Rightarrow \overline{AB} \cong \overline{DE}$, and it can happen that $\triangle ABC \cong \triangle DEF$ but $\neg(\triangle ABC \cong \triangle EFD)$.

Congruence of triangles.

Triangles consist of six measurable pieces (three line segments and three angles), which can be used to establish a notion of constancy of shape in triangles, which is important in geometry. We describe below some rules that allow us to, in essence, uniquely determine the shape of a triangle by looking at a specific subset of its pieces.

Axiom 8 (SAS Congruence). If triangles $\triangle ABC$ and $\triangle DEF$ have two congruent sides and a congruent included angle (meaning the angle between the sides in question), then the triangles are congruent. In particular, if $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.

Other congruence rules about triangles follow from the above: the ASA and SSS rules. However, their proofs are less interesting than other problems about triangles, so we leave them for the homework.

Axiom 9 (ASA Congruence). If two triangles have two congruent angles and a corresponding included side, then the triangles are congruent.

Axiom 10 (SSS Congruence). If two triangles have three sides congruent, then the triangles are congruent.

Homework problems

1. (Parallel and Perpendicular Lines) Part of the spirit of Euclidean geometry is that parallelism and perpendicularity are special concepts; Theorem 6, for example, is generally considered part of the heart of Euclidean geometry. For this problem, prove the following theorems presented in the First Theorems section, using only the information from the Basic Objects and First Postulates sections. Axiom 5 will be of key importance.
    a. Study the proof of Theorem 2 and draw a diagram that illustrates it.
    b. Study the proof of Theorem 3.
    c. Prove Theorem 5.
    d. Prove Theorem 6.
    e. Prove Theorem 7.
2. Complete the proof of Theorem 8, about sum of angles of a triangle.
3. What is the sum of angles of a quadrilateral? of a pentagon?
4. Prove the SSS rule of congruence for triangles.
5. Notice that SSA and AAA are not listed as congruence rules.
    a. Describe a pair of triangles that have two congruent sides and one congruent angle but are not congruent triangles.
    b. Describe a pair of triangles that have three congruent angles but are not congruent triangles.
6. Prove that the following two properties of a triangle are equivalent:
    a. All sides have the same length.
    b. All angles are $60°$

A triangle satisfying these properties is called equilateral.

7. A triangle in which two sides are congruent is called isosceles. Such triangles have many special properties.
    a. Let $\triangle ABC$ be an isosceles triangle, with $\overline{AB} \cong \overline{BC}$. Suppose D is a point on AC such that $\overline{AD} \cong \overline{DC}$ (such point is called midpoint of the segment). Prove that then, $\triangle ABD \cong \triangle CBD$ and deduce from this that $\angle DBA \cong \angle DBC$, and $\angle A \cong \angle C$. What can we say about $\angle ADB$?
    b. Conversely, show that if $\triangle ABC$ is such that $\angle A \cong \angle C$, then $\triangle ABC$ is isosceles, with $\overline{AB} \cong \overline{BC}$.