November 24, 2024 Math 8

Handout 12: Euclidean geometry 1: Axioms.

Geometry: definitions, axioms, and theorems

Euclidean geometry is a way to obtain an exact description of various geometric properties of various figures in the plane. What is a figure? Figures are understood as **sets** of **points**; we will use capital letters for points and write $P \in m$ for "point P belongs to figure m", or "figure m contains point P". A point is the simplest geometric object and is so basic that it cannot be explained in terms of even simpler notions; everyone is believed to know what a "point" is. In addition, there are some other basic notions that are not be defined and understood as elementary. These elementary notions, or **primitives** of planar Euclidean geometry include **point**, **line**, **plane**, **distance** and **angle measure**, and **set**. Instead, we can state some basic properties of these undefined objects; these basic properties are usually called **POSTULATES** or **AXIOMS** of Euclidean geometry, and these properties can be used to uniquely identify the respective objects. Using these basic properties, we can then apply rules of logic to deduce another property, which we call proving a theorem. **All results in Euclidean geometry should be proven by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable! In addition to rules of logic, in proving geometric theorems we will also use all the usual properties of real numbers, equations, inequalities, and alike. And, of course, geometrical constructs, or diagrams.

Note that we will make diagrams helping us understand relations between points, lines, angles, etc. Diagrams are never perfect (and do not have to be) because we use them only as aid to our thought process. Unlike diagrams, statements made in geometry are exact, and thus require more care to demonstrate than just a picture. For example, it is impossible to draw two precisely parallel lines; we can only "pretend" that their finite segments represent them in a picture, but we have to use axioms (and theorems already proven) to demonstrate additional properties. On its own, a picture is useful not only as illustration to help understand our argument, but also as a means of developing an argument by using various supplementary constructs. For your enjoyment, take a look at the book which gave rise to Euclidean geometry and much more, Euclid's Elements, dated about 300 BC, and used as the standard textbook for the next 2000 years. Nowadays it is available online at http://math.clarku.edu/~djoyce/java/elements/toc.html.

Geometry: Basic objects

These basic objects and notions are the basis of all our later constructions: we will define and discuss all other objects in terms of these primitives. We do not need (and cannot give) definitions for these basic objects, and consider them self-evident, as well as some of their properties.

- Points (denoted by upper-case letters: $A, B, \ldots$) can be said to have zero "size" in any respect;
- Lines (denoted by lower-case letters: $l, m, \ldots$): infinite in both directions and split the plane into "half-planes";
- Distances: for any two points A, B, there is a non-negative number AB, called the distance between A, B. The distance is zero **if and only if** points exactly coincide.

Note that one can measure distances with a ruler and angles with a protractor, but only as precisely (or imprecisely) as the tool allows. In geometry, however, these measures are considered **exact numbers**.

We will also frequently use words "between" when describing relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$). We do not give full list of axioms for these notions; it is possible, but rather boring.

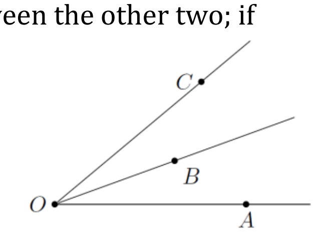
Having these basic notions, we can now define more objects. Namely, we can give definitions of

- interval, or line segment (notation: $\overline{AB}$): set of all points on a line $\overleftrightarrow{AB}$ which are between A and B, together with points A and B themselves. The segment length (denoted as AB or $|AB|$) is the distance between A and B.
- ray, or half-line (notation: $\overrightarrow{AB}$): set of all points on the line $\overleftrightarrow{AB}$ which are on the same side of A as B (Note that we have not defined the concept "on the same side" but will be using it in the future).
- angle (notation: $\angle AOB$): figure consisting of two rays ($\overrightarrow{OA}$ and $\overrightarrow{OB}$) with a common vertex (O). For any angle $\angle AOB$, there is a non-negative real number $m\angle AOB = \widehat{AOB}$, called the measure of this angle.
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points. We also say that every line is parallel to itself (it is a rather convenient convention, which will make our lives easier – the intuition here is that parallel lines have the same "direction"). Also, the angle between two parallel rays is zero.

Geometry: First Axioms

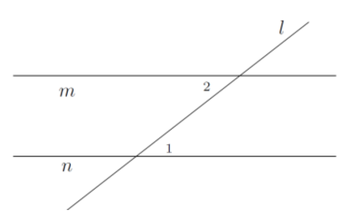
Axiom 1. For any two distinct points A, B, there is exactly one line to which both these points belong. (This line is usually denoted $\overleftrightarrow{AB}$). In other words, two distinct points are sufficient (and necessary) to specify a line.

Axiom 2. If distinct points A, B, C, are on the same line, exactly one is between the other two; if point B is between A and C, then $AC = AB + BC$.



Axiom 3. If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to 180.

Axiom 4. Let line l intersect lines m, n and angles $\angle 1, \angle 2$ are as shown in the figure (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.



In addition, we will assume that given a line l and a point A on it, for any positive real number d, there are exactly two points on l at distance d from A, on opposite sides of A, and similarly for angles: given a ray and angle measure, there are exactly two angles with that measure having that ray as one of the sides.

Homework problems

All problems below are important – please try to finish them all! Note that you may use all results that are presented in the previous sections. This means that you may use any theorem if you find it a useful logical step in your proof. The only exception is when you are explicitly asked to prove a given theorem, in which case you must understand how to draw the result of the theorem from previous theorems and axioms. Each step of your proof must be based on some previous, already proven, statement or an axiom.

1. It is important that you know some geometry notation.
   a. What does the symbol ∥ mean? How do you pronounce it? How would you read "$a \parallel b$"?
   b. What does the symbol ⊥ mean? How would you read "$a \perp b$"?
   c. Suppose you have two points X and Y. What is the difference between $\overline{XY}$, $\overrightarrow{XY}$, $\overleftrightarrow{XY}$? What are each of these things called?
   d. Given three points E, F, G, what does $|EF| + |FG|$ mean?
   e. Given four points A, B, C, D, what does $m\angle ADC + m\angle BDC$ mean? If I tell you $m\angle ADC + m\angle BDC = 180$, does that tell you any information about $m\angle ADC$ or $m\angle BDC$?
   f. What does the symbol △ mean? For example, if A, B, and C are points, what is $\triangle ABC$?
2.
   a. What is a proof? Give an example. Can you come up with an example that is not about geometry?
   b. What is an axiom? Give an example. Can you come up with an example that is not about geometry?
3. In this problem, you will make diagrams. Part of the purpose of this exercise is so that, when you think about geometry, the pictures in your notes or in your mind aren't all just the diagrams I draw out for you in class or on classwork sheets. You have to be able to draw or visualize configurations of lines other than the way they're set up in axiom 4, for example.
   a. Given lines a, b, c, is it possible that $a \parallel b$ and $\neg(b \parallel c)$ but $a \parallel c$? Draw a diagram and then explain your reasoning on how to answer this question ("explain" means, of course, in writing).
   b. Suppose we have parallel lines $l \parallel m$. Let A, B, C be points on l, with B between $A,\ C$,. Let X, Y, CZ be points on m, with Y between X, Z. Is it possible for lines $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, $\overleftrightarrow{CZ}$ to all intersect at one point? Draw a diagram of what this might look like.
   c. Consider the diagram you drew in the previous part, with the lines l, m and the six points, and the three cross-lines that intersect at a point. Now consider the lines $\overleftrightarrow{AZ}$, $\overleftrightarrow{CX}$. Do these two lines intersect at a point on Y? Draw a diagram where this is the case, and then draw a second diagram where this is not the case.
   d. Draw a rectangle that's not a square and draw it so that one of the bases is horizontal. Then draw one of the rectangle's diagonals. Notice that, of the two right angles formed at the rectangle's base, the rectangle's diagonal splits one of those angles into two smaller angles. Which of the two angles is bigger - the one below the diagonal, or the one above the diagonal? Draw a second rectangle where the opposite relation holds true (for example, if the lower angle was bigger in your first rectangle, draw a second rectangle where the lower angle split by the diagonal is smaller).
4. Can you formulate Axiom 4 without referring to the picture (i.e. without using any statement such as "angles $\angle 1, \angle 2$ are as shown in the figure"? You will have to introduce a number of points and have very clear notations.

5. The following logic and geometric statements come in equivalent pairs. Each logic statement has exactly one geometric statement that is equivalent to it. Match these statements into their equivalent pairs, with an explanation of why the pairs you chose are equivalent. [Note: the quantifier $\exists!$ Stands for "there exists a unique. . . ", and $\emptyset$ is an empty set.]

Geometric statements:

a. For any two distinct points there is a unique line containing these points.
b. Given a line and a point not on the line there exists a unique line though the given point that is parallel to the given line.
c. If two lines are parallel and another line intersects one of them, then it intersects the other one as well.
d. If two lines are parallel to the same line, then they are parallel to each other

Logic statements:

a. $\forall l, \forall m$ such that $l \parallel m$: $[\forall m, (n \cap l \neq \emptyset) \Rightarrow (n \cap m \neq \emptyset)]$
b. $\forall A, \forall B$ such that $A \neq B$: $[\exists!\ l, (A \in l) \wedge (B \in l)]$
c. $\forall l, \forall m$: $[\exists n$ such that $(n \parallel l) \wedge (n \parallel m) \Rightarrow (l \parallel m)]$
d. $\forall l, \forall A$ such that $A \notin l$: $[\exists!\ m, (A \in m) \wedge (m \parallel l)]$

Euclid's "Elements": Definitions

1. A point is that which has no part.
2. A line is breadthless length.
3. The ends of a line are points.
4. A straight line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The edges of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight lines on itself.
8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.
10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.
16. And the point is called the center of the circle.
17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.
18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.
22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Euclid's "Elements": Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Euclid's "Elements": Common notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.