November 17, 2024 Math 8

Handout 11: Logic: Proofs and Quantifiers.

Existential Quantifier

To write statements of the form "There exists an x such that ...", we use existential quantifier,

$$\exists x \in A: (\textit{some statement depending on } x).$$

Here, A is some set of values of x, from which one can select a value which makes the statement true or false. Note that following the quantifier, you must have a statement, i.e. something that can be true or false. Usually, it is some equality or inequality. You can't write there an expression which gives numerical values – it makes no sense (for example, $\exists x \in \mathbb{Z}: (x^2 + 1)$ makes no sense).

Example: $\exists x \in \mathbb{Z}: x^2 = 4$. Indeed, $x = 2$ is a value for which the statement is true; so it is for $x = -2$, but you only need one!

Exercise 1. Is the following statement true or false? Present a proof of your answer.

- $\exists x \in \mathbb{Z}: x^2 - 4x - 5 = 0$
- $\exists x \in \mathbb{Z}: x^2 = 5$
- $\exists x \in \mathbb{Q}: x^2 = 5$
- $\exists x \in \mathbb{Q}: 2x^3 + x^2 - 4x - 2 = 0$

Universal Quantifier

To write statements of the form "For all values of x we have...", we use the universal quantifier,

$$\forall x \in A: (\textit{some statement depending on } x).$$

Here, A is some set of values of x, from which one can select a value which makes the statement true or false. Again, following the quantifier, you must have a statement, i.e. something that can be true or false.

Example: $\forall x \in \mathbb{Q}: x^2 > 0$. Indeed, a square of any rational number cannot be negative (however, there are so-called complex numbers for which this is not true!).

Exercise 2. Is the following statement true or false? Present a proof of your answer.

- $\forall x \in \mathbb{Z}: x^2 - 4x - 5 = 0$
- $\forall x \in \mathbb{Q}: 2x^2 + 5x + 2 > 0$
- $\forall x \in \mathbb{N}: x^2 > x$

Proofs with Quantifiers

To prove a statement $\exists x \in A: P(x)$, it suffices to give one example of x for which the statement $P(x)$ is true. It is sufficient to verify that the statement is true just for that value x, but it is not necessary to explain how you found this value, nor is it necessary to find how many such values there are.

Example: to prove $\exists x \in \mathbb{Q}: x^2 = 9$, take $x = 3$; then $x^2 = 9$.

To prove a statement $\forall x \in A: Q(x)$, you need to give an argument which shows that for any $x \in A$, the statement $Q(x)$ is true. Considering one, two, or one thousand examples where $Q(x)$ is true is not enough!!!

Example: to prove $\forall x \in \mathbb{Q}: x^2 + 2x + 4 > 0$, we could argue as follows. Let x be an arbitrary rational number.

Then $x^2 + 2x + 4 = (x + 1)^2 + 3$. Since a square of a rational number is always non-negative, $(x + 1)^2 \geq 0$, it follows that $x^2 + 2x + 4 = (x + 1)^2 + 3 \geq 0 + 3 > 0$. Note that this argument works for any x; it uses no special properties of x except that x is a real number.

De Morgan Law for Quantifiers

Assuming that A is a nonempty set,

$$\neg(\forall x \in A: P(x)) \Leftrightarrow (\exists x \in A: \neg P(x))$$

$$\neg(\exists x \in A: P(x)) \Leftrightarrow (\forall x \in A: \neg P(x))$$

For example, negation of the statement "All flowers are white" is "There exists a flower which is not white", or in more human language, "Some flowers are not white".

Exercise 3. Here is another one of Lewis Carroll's puzzles. As before, (a) write the obvious conclusion from given statements and (b) justify the conclusion by writing a chain of arguments which leads to it.

- No one subscribes to the Times, unless he is well educated.
- No hedgehogs can read.
- Those who cannot read are not well educated.

It may be helpful to write each of these as a statement about some particular being X, e.g. "If X is a hedgehog, then X can't read."

Exercise 4. Prove that $\sqrt{2}$ cannot be rational: it cannot be a ratio $\frac{p}{q}$ of two integer numbers, p and q.

Hint: assume that it can, and that p and q do not have common factors (if there are, you can always cancel them). Do either p or q (or both) have to be even?

Homework problems

1. The following statement is sometimes written on highway trucks: If you can't see my windows, I can't see you. Let's use A for "you can see my windows" and B for "I can see you".
    a. Can you write an equivalent statement without using word "not"?
    b. Rephrase the statement using "necessary" and "sufficient".
2. Write the following statements using quantifiers: (You can use letter B for the set of all birds, and notation $F(x)$ for statement "x can fly" and $L(x)$ for "x is large").
    a. All birds can fly
    b. Not all birds can fly
    c. Some birds can fly
    d. All large birds can fly
    e. Only large birds can fly
    f. No large bird can fly
3. Write the following statements using logic operations and quantifiers:
    a. All mathematicians love music
    b. Some mathematicians don't like music
    c. No one but a mathematician likes music
    d. No one would go to John's party unless he loves music or is a mathematician

Please use the following notation:

- $P =$ set of all people
- $M(x) = x$ is a mathematician
- $L(x) = x$ loves music
- $J(x) = x$ goes to John's party

4. Write each of the following statements using only quantifiers, arithmetic operations, equalities and inequalities. In all problems, letters x, y, z stand for variables that take real values, and letters $m, n, k, \ldots$ stand for variables that take integer values.
    a. Equation $x^2 + x - 1$ has a solution
    b. Inequality $y^3 + 3y + 1 < 0$ has a solution
    c. Inequality $y^3 + 3y + 1 < 0$ has a positive real solution
    d. Number 100 is even
    e. Number 100 is odd
    f. For any integer number, if it is even, then its square is also even.
5. Prove that for any integer number n, the number $n(n+1)(2n+1)$ is divisible by 3. Is it true that such a number must also be divisible by 6? You can use without proof the fact that any integer can be written in one of the forms $n = 3k$ or $n = 3k + 1$ or $n = 3k + 2$, for some integer k.
6. Prove that $A \Rightarrow C$ provided you are given the following true statements:
    a. $(A \wedge B) \Rightarrow C$
    b. $B \vee D$
    c. $C \vee \neg D$
7. A function $f(x)$ is called monotonic if $(x_1 < x_2) \; =) \; (f(x_1) < f(x_2))$. Prove that a monotonic function can't have more than one root. [Hint: assume that it has two distinct roots and derive a contradiction.]